Connor Colombe
Putnam Seminar

Q.4) For positive integers m and n, let $f(m,n)$ denote the number of n-tuples $(x_1, x_2, \ldots, x_n)$ of integers such that $|x_1| + |x_2| + \ldots + |x_n| \leq m$. Show that $f(m,n) = f(n,m)$.

Solution:

We will first develop an expression for the number of n-tuples $(x_1, x_2, \ldots, x_n)$ of integers such that

$$|x_1| + |x_2| + \ldots + |x_n| = r$$

where r is a positive integer of value at most m, then we will sum over these expressions to obtain an expression for $f(m,n)$. The same logic also provides a similar formula for $f(n,m)$. We will then manipulate each of these expressions so that they are in a manageable form and then prove equality using a combinatorial argument.

Let r be a positive integer of value at most m. Let $g_n(r)$ be the number of n-tuples $(x_1, x_2, \ldots, x_n)$ of integers such that $|x_1| + |x_2| + \ldots + |x_n| = r$. The main difficulty in determining a formula for $g_n(r)$ is that some of the x_i may be zero. To find some order in this, consider counting the number of n tuples in which exactly k entries are zero and whose absolute value sum is r.

There are $\binom{n}{k}$ ways to choose which entries are set to zero. Now we need to distribute a total sum of r over the remaining $n-k$ absolute valued entries. This can be done by first distributing one unit to each of the $n-k$ entries, then distributing the remaining $r-(n-k)$ units with no restrictions which can be done in $\left(\!\binom{n-k}{r-(n-k)}\!\right) = \binom{r-1}{r-n-k}$ ways. Lastly, we need to account for the signs of each non-zero entry, they can be negative or positive and it doesn't alter the sum, therefore there are 2^{n-k} ways to pick the signs. Putting it all together, there are

$$\binom{n}{k}\binom{r-1}{r-n-k}2^{n-k} \tag{1}$$

n-tuples of integers that have absolute value sum r with exactly k zero entries, summing over all values of k, we have the total number of n-tuples that have absolute value sum equal to r is

$$g_n(r) = \sum_{k=0}^{n} \binom{n}{k}\binom{r-1}{r-n-k}2^{n-k}. \tag{2}$$

Now, to obtain an expression for $f(m,n)$, we sum over the possible values of $r = 1, 2, \ldots, m$ to obtain

$$f(m,n) = \sum_{r=1}^{m}\sum_{k=0}^{n} \binom{n}{k}\binom{r-1}{r-n-k}2^{n-k}. \tag{3}$$

Since we are summing over a finite number of terms, we can swap the order of summation. Then, we can factor out some terms and utilize the "hockey stick identity" from Pascal's triangle [1] on the inner sum.

$$
\begin{aligned}
f(m,n) &= \sum_{r=1}^{m}\sum_{k=0}^{n}\binom{n}{k}\binom{r-1}{r-n-k}2^{n-k} \\
&= \sum_{k=0}^{n}\sum_{r=1}^{m}\binom{n}{k}\binom{r-1}{r-n-k}2^{n-k} \\
&= \sum_{k=0}^{n}\binom{n}{k}2^{n-k}\sum_{r=1}^{m}\binom{r-1}{r-n-k} \\
&= \sum_{k=0}^{n}\binom{n}{k}2^{n-k}\binom{m}{m-n+k}
\end{aligned}
$$

Now, taking advantage of the symmetry of the binomial coefficient, we can write $\binom{m}{m-n+k}$ as $\binom{m}{n-k}$ and we are left with the overall expression

$$
f(m,n) = \sum_{k=0}^{n}\binom{n}{k}\binom{m}{n-k}2^{n-k}. \tag{4}
$$

The same argument presented tells us

$$
f(n,m) = \sum_{k=0}^{m}\binom{m}{k}\binom{n}{m-k}2^{m-k} \tag{5}
$$

and it remains to show these two sums are equal. We will do so with a combinatorial proof. Consider a set of $n+m$ people n of which are wearing blue shirts and m of which are wearing red shirts. The sum in (4) is the number of ways to pick a committee of size n and then any sized subcommittee from the people wearing red shirts in the committee.

The sum in (5) picks a subset of size m that will *not* be on the committee (and in doing so determines the n people that will be on the committee) and then chooses any sized subcommittee of the remaining people wearing red shirts to be on the subcommittee. Since theses two sums are counting the same thing, they are equal.

Therefore $f(n,m) = f(m,n)$ as desired.

[1] See https://en.wikipedia.org/wiki/Hockey-stick_identity