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Putnam Seminar

Q.6) Define a *selfish set* to be a set which has its own cardinality (number of elements) as an element. Find, with a proof, the number of subsets of $\{1, 2, \ldots, n\}$ which are minimal selfish sets, that is, selfish sets none of whose proper subsets is selfish.

Solution:
let's first introduce some notation. Let A_k where $1 \leq k \leq n$, be the set of all minimal selfish sets of size k with elements from $N = \{1, 2, \ldots, n\}$ and let M_n be the total number of minimal selfish sets formed from subsets of N.

With this notation established, notice that by definition of a selfish set, if $S \in A_k$ then $k \in S$. Moreover, we claim that if $\alpha \in S$ and $\alpha \neq k$ then $\alpha > k$. To see why, if $\alpha < k$, and $|S| = k$, when we can find a subset of size α in S that includes α, violating our assumption that S was a minimal selfish set.

Therefore, every $S \in A_k$ has the form $S = \{k, \alpha_1, \ldots, \alpha_{k-1}\}$ where $\alpha_i > k$ for all $i = 1, \ldots, k-1$. Are there any more restrictions on the α_i? No. To see why this is true notice that we cannot form a selfish subset of cardinality higher than k regardless of the values of α_i's since any subset of S has cardinality at most k. It remains to count the number of minimal selfish subsets of size k. Note that we have $k-1$ spots/α's to fill and we have $n-k$ remaining values to choose from, therefore

$$|A_k| = \binom{n-k}{k-1}. \tag{1}$$

Summing over $k \in N$ we have an expression for the total number of minimal selfish sets M_n.[1]

$$M_n = \sum_{k=1}^{n} \binom{n-k}{k-1} \tag{2}$$

Re-indexing, and using footnote 1 we can write (2) as

$$M_n = \sum_{k=0}^{\infty} \binom{n-1-k}{k} \tag{3}$$

Experimenting with some values, we notice the pattern that M_n seems to align with F_n, the n-th Fibonacci number starting with $F_1 = 1,\ F_2 = 1$. To finish, we will establish a fact about the Fibonacci numbers and then use it prove $F_n = \sum_{k=0} \binom{n-1-k}{k}$ by a combinatorial proof.

[1] We could technically sum to infinity since $\binom{n}{k} = 0$ for all $k > n$

Lemma 1. *I claim the F_n is the number of ways to tile an $(n-1) \times 1$ rectangular board with (2×1) rectangular and (1×1) square pieces.*

Proof. We proceed by strong induction. For $n = 1$, we note there is only one way to tile a 0×1 board which is F_1. For $n = 2$ there again is only one way to time the unit board which agrees with F_2.

Now we make the inductive hypothesis that this holds for all natural numbers $n \leq k$. Consider an rectangular board of size $k \times 1$. We can either place a rectangular or square piece at the leftmost position, each producing a remaining board size of $k-1$ and $k-2$ respectively, but by our inductive hypothesis, these can be tiled in F_k and F_{k-1} ways receptively. Since our first tile placements were mutually exclusive moves, there are $F_k + F_{k-1} = F_{k+1}$ ways to tile the size k board, as desired. □

With this lemma established, we now want to show that there is another way to count the tilings of a size $(n-1) \times 1$ board. That is, we wish to show

$$M_n = \sum_{k=0}^{\infty} \binom{n-1-k}{k}$$

is also the number of ways to tile a board of size $(n-1) \times 1$. One way to see why this true is to notice each nonzero term, $\binom{n-1-k}{k}$, in the sum, is the number of ways to tile an $n-1$ length board using *exactly* k of the (2×1) sized rectangular pieces, and the rest (1×1) squares. To see this, note that any particular arrangement of k of the (2×1) pieces on a $(n-1) \times 1$ board can be encoded in k positions on a $(n-1-k) \times 1$ board. Simply collapse each (2×1) piece used into a (1×1) piece. Similarly we can expand any such choice of k postions on a $(n-1-k) \times 1$ board to a unique placement of (2×1) pieces on an $(n-1) \times 1$ board. If we now sum over the number of (2×1) pieces to be used in the tilings, we acheive

$$F_n = \sum_{k=0}^{\infty} \binom{n-1-k}{k}$$

and finally we have proven

$$\boxed{M_n = F_n}$$

where $F_1, F_2 = 1$