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Putnam Seminar

Q.8) Find a combinatorial explanation for the algebraic identity

$$\binom{\binom{n}{2}}{2} = 3\binom{n+1}{4}$$

Solution:

Let us first decipher the left-hand side. $\binom{n}{2}$ is the number of two element subsets from a set of size n. Therefore $\binom{\binom{n}{2}}{2}$ is the number of ways two pick two, distinct two element subsets from a set of size n. Now we have two cases to consider, the two subsets we chose are disjoint, in which case their union is of the form $\{A, B\} \cup \{B, C\} = \{A, B, C, D\}$, or they share a single common element, in which case their union is of the form $\{A, B\} \cup \{A, C\} = \{A, B, C\}$, lets call the former subsets of the 'first' type and the latter subsets of the 'second' type.

Given a particular subset of the 'first' type, $S_1 = \{A, B, C, D\}$ there are three choices of two element subsets whose union give us S_1, namely,

$$\{A, B\} \cup \{C, D\} = \{A, B, C, D\}$$
$$\{A, C\} \cup \{B, D\} = \{A, B, C, D\}$$
$$\{A, D\} \cup \{B, C\} = \{A, B, C, D\}$$

Similarly, given a particular subset of the 'second' type, $S_2 = \{A, B, C\}$ there are three choices of two element subsets whose union give us S_2, which are,

$$\{A, B\} \cup \{B, C\} = \{A, B, C\}$$
$$\{A, C\} \cup \{B, C\} = \{A, B, C\}$$
$$\{A, B\} \cup \{A, C\} = \{A, B, C\}$$

Anthropomorphizing, if we have a set of n people, then $\binom{\binom{n}{2}}{2}$ is three times the number of ways to pick a subset of 3 or 4 of them.

Inspired by this observation, let us now determine the number of ways to pick a subset of 3 or 4 people from a pool of n people. A neat way to think about this is to add a fictitious 'invisible' person to the pool of n people. Now when we pick a group of size 3, we really chose a group of size 4 with the invisible person chosen. With this modification, it follows that the number of ways to pick a group of size 3 or 4 is the same as the number of ways to choose a group of size 4 including a possibly invisible person which is $\binom{n+1}{4}$. Multiplying by a factor of three to get, $3\binom{n+1}{4}$ both sides of our original equation now represent three times the number of ways to pick a group of size 3 or 4 from a set of n people.