These Putnam(ish) problems are, as you requested, about combinatorics (broadly construed).

1. Show that if p is prime then for all positive integers a and b,

$$\binom{pa}{pb} \equiv \binom{a}{b} \pmod{p}$$

2. Recall that a regular icosahedron is a convex polyhedron having 12 vertices and 20 faces; the faces are congruent equilateral triangles. On each face of a regular icosahedron is written a nonnegative integer such that the sum of all 20 integers is 39. Show that there are two faces that share a vertex and have the same integer written on them.

3. A round-robin tournament among $2n$ teams lasted for $2n-1$ days, as follows. On each day, every team played one game against another team, with one team winning and one team losing in each of the n games. Over the course of the tournament, each team played every other team exactly once. Can one necessarily choose one winning team from each day without choosing any team more than once?

4. For positive integers m and n, let $f(m,n)$ denote the number of n-tuples $(x_1, x_2, \ldots, x_n)$ of integers such that $|x_1| + |x_2| + \ldots + |x_n| \leq m$. Show that $f(m,n) = f(n,m)$.

5. In Determinant Tic-Tac-Toe, Player 1 enters a 1 in an empty 3×3 matrix. Player 0 counters with a 0 in a vacant position, and play continues in turn until the 3×3 matrix is completed with five 1s and four 0s. Player 0 wins if the determinant is 0 and Player 1 wins otherwise. Assuming both players pursue optimal strategies, who will win and how?

6. Define a selfish set to be a set which has its own cardinality (number of elements) as an element. Find, with a proof, the number of subsets of $\{1, 2, \ldots, n\}$ which are minimal selfish sets, that is, selfish sets none of whose proper subsets is selfish.

7. Let S be a set of n distinct real numbers. Let A_S be the set of numbers that occur as averages of two distinct elements of S. For a given $n \geq 2$, what is the smallest possible number of distinct elements in A_S?

8. Find a combinatorial explanation for the algebraic identity

$$\binom{\binom{n}{2}}{2} = 3\binom{n+1}{4}$$