Putnam-group challenging Geometry problems

1. A rectangle $HOMF$ has sides $HO = 11$ and $OM = 5$. A triangle $ABC$ has $H$ as the intersection of its altitudes, $O$ as the center of its circumscribed circle, $M$ as the midpoints of $BC$, and $F$ as the foot of the altitude from $A$. What is the length of $BC$?

2. Let $d_1, d_2, \ldots, d_{12}$ be real numbers in the open interval $(1, 12)$. Show that there exist distinct indices $i, j, k$ such that $d_i, d_j, d_k$ are the side lengths of an acute triangle.

3. What is the maximum number of rational points that can be on a circle in $\mathbf{R}^2$ whose center is not a rational point? (A *rational point* is a point both of whose coordinates are rational numbers.)

4. Show that for any set of five points on a sphere there is a set of four of them that lie on a closed hemisphere.

5. Can an arc of a parabola inside a circle of radius 1 have length greater than 4?

6. A unit cube is positioned in $R^3$ in some orientation and projected projected onto the coordinate plane $\{x_1 = 0\}$. What is the largest possible area of this projection?

In the following two questions, $\theta_n$ is the measure of an angle which is one $n$th of a complete circle. Since it is famously true that $\theta_{34}$ can be constructed with straightedge and compass, but $\theta_{36}$ cannot be, we will feature these angles in the remaining questions. In both cases we present a construction and ask you to compute the measure of angle $PCB$.

7. A point $P$ lies inside the triangle $ABC$. The measures of the angles $PAC, PAB, PBA$, and $PBC$ are respectively $3\theta_{36}, \theta_{36}, 2\theta_{36}$, and $2\theta_{36}$. Compute the measure of angle $PCB$.

8. A point $P$ lies off the line segment $ABC$. The measures of the angles $PAB$ and $APB$ are respectively $5\theta_{34}$ and $9\theta_{34}$, and segment $AB$ is twice as long as segment $BC$. Compute the measure of angle $PCB$.

Hint: One of these last two prolems is almost, but not quite, doable and the other is almost, but not quite, un-doable!