## UT Putnam Prep Problems, Nov 2, 2016 (Go, Cubbies, Go!) NUMBER-THEORY PUTNAM PROBLEMS

1. Find all positive integers that are within 250 of exactly 15 perfect squares. (Note: A perfect square is the square of an integer; that is, a member of the set $\{0,1,4,9,16, \ldots\}$. We say " $a$ is within $n$ of $b$ " if $b-n \leq a \leq b+n$.)
2. Find the smallest positive integer $n$ such that for every integer $m$, with $0<m<1993$, there exists an integer $k$ for which

$$
\frac{m}{1993}<\frac{k}{n}<\frac{m+1}{1994}
$$

3. How many primes among the positive integers, written as usual in base 10, are such that their digits are alternating 1 s and 0 s , beginning and ending with 1 ?
4. A composite (positive integer) is a product $a b$ with $a$ and $b$ not necessarily distinct integers in $\{2,3,4, \ldots\}$. Show that every composite is expressible as $x y+x z+y z+1$, with $x, y$, and $z$ positive integers.
5. For a given positive integer $m$, find all triples $(n, x, y)$ of positive integers, with $n$ relatively prime to $m$, which satisfy $\left(x^{2}+y^{2}\right)^{m}=(x y)^{n}$.
6. Prove that, for any integers $a, b, c$ there exists a positive integer $n$ such that

$$
\sqrt{n^{3}+a n^{2}+b n+c}
$$

is not an integer.
7. Let $N$ be the positive integer with 2016 decimal digits, all of them being 1 , that is, $N=111 \ldots 111$ (2016 digits). Find the thousandth digit after the decimal point of $\sqrt{N}$.
8. Let $S$ be a finite set of integers, each greater than 1. Suppose that for each integer $n$ there is some $s \in S$ such that either $\operatorname{gcd}(s, n)=1$ or $\operatorname{gcd}(s, n)=s$. Show that there exists $s, t \in S$ such that $\operatorname{gcd}(s, t)$ is prime. [Here $\operatorname{gcd}(a, b)$ denotes the greatest common divisor of $a$ and $b$.]
9. Show that no four consecutive binomial coefficients can lie in an arithmetic progression:

$$
\binom{n}{r}, \quad\binom{n}{r+1}, \quad\binom{n}{r+2}, \quad\binom{n}{r+3}
$$

