That Long-Parabola problem

Since I know several people worked long and hard on the parabola question, I thought I'd illustrate a solution that requires the least algebra. We will show that there are parabolic arcs in a circle that are longer than twice the diameter, but they do have to be narrow, and in no case does the excess length amount to even one part in a thousand!

Consider the simple parabola $y = x^2$. A portion of it is included inside the circle $x^2 + (y-r)^2 = r^2$, to which it is tangent at the origin. In fact, the coordinates (x, y) of the points of intersection will satisfy both these equations, and hence also satsify $y + (y-r)^2 = r^2$, which simplifies to $y(1 + y - 2r) = 0$; thus the curves meet at the origin and, if $2r - 1 > 0$, at the two points having $y = 2r - 1$ (whose x coordinates we will write as $\pm a$, where $a = \sqrt{2r-1} > 0$). So the entirety of the parabola between the points $(-a, a^2)$ and $(+a, a^2)$ lies inside this circle of radius $r = (a^2+1)/2$. Using Calculus (and symmetry) the length of that curve is twice the value of the integral

$$I(a) = \int_0^a \sqrt{1+4x^2}\, dx$$

If we could show that for some value of a, the value of the $I(a)$ exceeds $a^2 + 1$, then the length of the curve would be more than $2(a^2+1) = 4r$; scaling the whole picture by a factor of r would then give a parabolic arc of length larger than 4 inside a circle of radius 1. So the whole question comes down to asking when, if ever, we have $I(a) > a^2 + 1$.

At this point you have a choice: do you just want to prove the desired inequality, or do you want to learn more about the parabolas and their lengths?

For the first goal, we are nearly there already: obviously $\sqrt{1+4x^2}$ is approximately $2x$ for large values of x, and we even have an inequality in the direction we want: since $\sqrt{1+4x^2} > 2x$, we deduce $I(a) > \int_0^a 2x\, dx = a^2$. That's (obviously) not enough to show $I(a) > a^2+1$ but it's clear we are almost accurate enough in our estimate of the integral. I will show below that the next most-dominant term in an approximation for the integrand is $\sqrt{1+4x^2} \approx 2x + \frac{1}{4x}$ but even though it's more accurate, the inequality works the other way this time. Still, if the integrand is *approximately* $2x + \frac{1}{4x}$ for large x, and *larger* than $2x$ itserlf, then surely it will be larger than, say, $y = 2x + \frac{1}{8x}$ if x is large enough. Indeed, $4x^2 + 1$ will be greater than $y^2 = 4x^2 + \frac{1}{2} + \frac{1}{64x^2}$ as long as $64x^2 > 2$; for example this is true for every $x > 0.2$. Thus for large a we have

$$\begin{aligned} I(a) &= I(0.2) + \int_{0.2}^a \sqrt{1+4x^2}\, dx > I(0.2) + \int_{0.2}^a \left(2x + \frac{1}{8x}\right) dx \\ &= I(0.2) + (a^2 - 0.04) + \frac{\log(a)}{8} - \frac{\log(0.2)}{8} \\ &= a^2 + \frac{\log(a)}{8} + C \end{aligned}$$

for some constant C. Clearly this will exceed $a^2 + 1$ for sufficiently large a. (Simple estimates show $C = I(0.2) - 0.04 + \log(5)/8 > 0.2 - 0.04 + 1/8 > 1/4$, so we will surely have $I(a) > a^2 + 1$ for $a > e^6$, for example.)

This solves the Putnam problem, which asked merely for the existence of such an a. We can be somewhat more precise and study what happens as the parameter a varies.

We could begin by using a better approximation to our integrand. Writing $\sqrt{1+4x^2} = (2x)\sqrt{1+\frac{1}{4x^2}}$ and using the Maclaurin series $\sqrt{1+t} = 1 + (t/2) - (t^2/8) + \dots$, we may approximate the integrand as $2x + \frac{1}{4x} - \frac{1}{64x^3} + \dots$, and then as in the previous paragraph we determine the rate of growth of $I(a)$ as being $a^2 + \log(a)/4 + C$ for suitable constants C. We leave to the reader a careful analysis (using the remainder estimate for Taylor series). In fact, if $f(x) = x^2 + \log(x)/4$ then $I(a) > f(a) + \log(2)/2 + 1/8$ for all a, with the difference approaching 0 as $a \to \infty$.

Alternatively, rather than using estimates for the integrand, we may evaluate the integral in closed form using standard Calculus techniques: it's

$$\frac{a}{2}\sqrt{1+4a^2} + \frac{1}{4}\log\left(2a + \sqrt{1+4a^2}\right)$$

From here it is reasonable to use Taylor series to estimate $I(a)$ itself (rather than the integrand $\sqrt{1+4x^2}$). Writing the first term as $a^2\sqrt{1+\frac{1}{4a^2}}$, a Taylor-series estimate renders this as $a^2 + \frac{1}{8} - \frac{1}{128a^2} + \dots$. Likewise the argument to the logarithm in the second term may be written as $4a(1 + (1/16a^2) + \dots)$; using the Taylor series for the logarithm then allows us to estimate the second term as $\log(4a)/4 + \frac{1}{16a^2} + \dots$. In other words, we may obtain a Taylor series estimate for $I(a) - (a^2 + \log(4a)/4)$ which begins $\frac{1}{8} + \frac{7}{128a^2} + \dots$. This is obviously positive for all large a, so that for all large a we have $I(a) > a^2 + \log(4a)/4$, and in particular for all large a we will have $I(a) > a^2 + 1$.

In fact we should expect $I(a) > a^2 + 1$ as soon as $\log(4a)/4 + 1/8 > 1$ since the next term in the Taylor series has a positive coefficient. That inequality is equivalent to $a > (1/4)e^{7/2} \approx 8.279$, and indeed we find with some numerical work that $I(a) > a^2 + 1$ for all $a > 8.275$.

Clearly, $I(a) - a^2$ is an increasing function, since its derivative $\sqrt{1+4a^2} - 2a$ is positive. So once we have $I(a) > a^2 + 1$ for one value of a, the inequality also holds for all larger values of a. This if any of the parabolas is long enough, then all the "sharper" parabolas are long enough too!

The critical value $a = 8.275$ corresponds to a circle of radius $r = (a^2 + 1)/2 = 34.74$. When scaled to a circle of radius 1, we then have parabolic arcs of length greater than 4 for all narrower parabolas than this one, although the scaled lengths decrease towards 4 again past $a = 13.6815$ or so. (This corresponds to a circle of radius roughly 94.09; that scaled parabola has the largest possible length, approximately 4.0027.)

At this point I'm just having fun with the construction; I encourage you to do likewise, but I do stress that the original Putnam problem can be solved without such details!