While working the first question this week, we needed to optimize a function of three variables, subject to some constraints. The function to be optimized was symmetric with respect to the three variables (that is, interchanging any two of them leaves the value of the function unchanged) and the same was true of the constraints. The claim was then made that the optimal point would have to have equal values for the three variables.

This is not true in general, and I can illustrate this with an example. Let

$$f(x, y, z) = (xy)^2 + (yz)^2 + (zx)^2$$
 $g(x, y, z) = x + y + z$

and consider the problem of minimizing the value of f subject to the constraint that g = 1.

Since f is a sum of squares, its values are nonnegative everywhere, and thus the minimal possible value of f anywhere is 0. Indeed, we can determine all the points where f = 0: this can only occur if all three of the squares is equal to zero. But if for example xy = 0 then either x or y must be zero. Since the other squares must also vanish, we deduce that the necessary and sufficient condition to ensure that f = 0 is that two of the coordinates equal zero, that is, the point (x, y, z) must lie on one of the axes. Given the additional constraint that g = 1, we must have the third coordinate equal 1. In other words, the minimum value of f on the surface where g = 1 is zero, and there are three points where this occurs: at (1,0,0), (0,1,0), and (0,0,1).

By contrast, the only point having equal coordinates which satisfies the constraint is the point (1/3, 1/3, 1/3); this point is a critical point but it's not even a *local* minimum! (Even better: it's not even a classical saddle point; it's an example of what is called a "monkey saddle" point).

What happens in this example is true in general: if there is a symmetry to the function and to the constraints as well, then there will be the same symmetry to the *set of* critical points. In the special case that the critical point is unique, then yes, that will mean the same symmetry will apply to the coordinates themselves of that point. But as in this example, there can be multiple critical points, permuted amongst themselves by the symmetries. (Our example has only these four critical points, three of them in one orbit under the symmetry group, the last fixed by the symmetries.)