M 393C/CSE 396 Homework 1

Due Thursday, September 15, 2016

1. Folland Section 7.1, problems 2,3, and 4 (copied below).

   7.2 The Fourier transform 213

   2. Let $f(x) = |x|^{-p}$ where $\frac{1}{2} < p < 1$. Show that f is in neither L^1 nor L^2, but that f can be expressed as the sum of an L^1 function and an L^2 function.
   3. Let $f(x) = 1$ if $-1 < x < 1$, $f(x) = 0$ otherwise.
      a. Compute $f * f$ and $f * f * f$.
      b. Let $f_\epsilon(x) = \epsilon^{-1} f(\epsilon^{-1} x)$ as in (7.3) and let $g(x) = x^3 - x$. Compute $f_\epsilon * g$ and check directly that $f_\epsilon * g \to 2g$ as $\epsilon \to 0$. (Note that $2 = \int f(x)\, dx$.)
   4. Let $f(x) = e^{-x^2}$ and $g(x) = e^{-2x^2}$. Compute $f * g$. (Hint: Complete the square in the exponent and use the fact that $\int e^{-x^2} dx = \sqrt{\pi}$.)

2. Suppose $f, g \in L^1$. Prove the following Fourier transform formulas:
   (a) If $f \in C^1(\mathbb{R})$ and $f'(x) \to 0$ as $|x| \to \infty$, then $\mathcal{F}\{f'\}(\xi) = (i\xi)\widehat{f}(\xi)$.
   (b) $\mathcal{F}\{f * g\}(\xi) = \widehat{f}(\xi) \cdot \widehat{g}(\xi)$

3. Show that for any $f \in L^2$ and any $\delta > 0$, there is a function g such that (i) g is of class $C^{(\infty)}$, (ii) g vanishes outside a finite interval, and (iii) $\|f - g\| < \delta$. Proceed by the following steps:
   - Let $F(x) = f(x)$ if $|x| < N$, $F(x) = 0$ otherwise. Show that $\|F - f\| < \frac{1}{2}\delta$ if N is sufficiently large
   - Show that $g = F * K_\epsilon$ does the job if ϵ is sufficiently small and K is given by $K(y) = C^{-1} e^{-1/(1-y^2)}$ for $|y| < 1$, $K(y) = 0$ for $|y| \geq 1$.

4. Prove that the *sinc* function, $\operatorname{sinc}(x) := \frac{\sin(\pi x)}{\pi x}$, $x \in \mathbb{R}$, belongs to $L^2(\mathbb{R})$, but not to $L^1(\mathbb{R})$.

5. Folland Section 7.2, problems 5 and 7 (copied below).

5. Suppose $g \in L^1$, $\int g(x)\,dx = 1$, and $\widehat{g} \in L^1$.
   a. Show that $\widehat{g}(\delta\xi) \to 1$ as $\delta \to 0$ for all $\xi \in \mathbf{R}$.
   b. Show that for any continuous $f \in L^1$,

$$\lim_{\delta \to 0} \frac{1}{2\pi} \int e^{i\xi x} \widehat{g}(\delta\xi) \widehat{f}(\xi)\, d\xi = f(x)$$

   for all x. What if f is only piecewise continuous? (Mimic the argument leading to (7.15), using the Fourier inversion theorem for g.)
6. Show that $\int_0^\infty x^{-1}|\sin x|\,dx = \infty$. (Hint: Show that $\int_{(n-1)\pi}^{n\pi} x^{-1}|\sin x|\,dx > 2/n$.)
7. Suppose that f is continuous and piecewise smooth, $f \in L^2$, and $f' \in L^2$. Show that $\widehat{f} \in L^1$. (Hint: First show that $\int(1+\xi^2)|\widehat{f}(\xi)|^2 d\xi$ is finite; then use the Cauchy-Schwarz inequality as in the proof of Theorem 2.3, §2.3.)