M 393C/CSE 396 Homework 2

Due Tuesday, October 11, 2016

- Folland Section 7.3, problems 8, 9, and 10 (copied below). There is a typo in problem 10: $f' + cf = 0$ should be $f'(x) + cxf(x) = 0$.

8. Suppose $f \in L^2(\mathbf{R})$, $\widehat{f}(\omega) = 0$ for $|\omega| > \Omega$, and $\lambda > 1$.
   a. As in the proof of the sampling theorem, show that

$$\widehat{f}(\omega) = \frac{\pi}{\lambda\Omega} \sum_{-\infty}^{\infty} f\left(\frac{n\pi}{\lambda\Omega}\right) e^{-in\pi\omega/\lambda\Omega} \quad \text{for} \quad |\omega| \leq \lambda\Omega.$$

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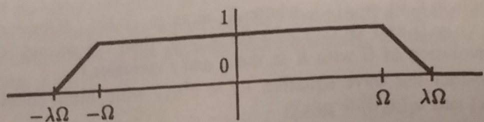
   b. Let $\widehat{g}_\lambda$ be the piecewise linear function sketched below. Show that the inverse Fourier transform of $\widehat{g}_\lambda$ is

$$g_\lambda(t) = \frac{\cos \Omega t - \cos \lambda\Omega t}{\pi(\lambda - 1)\Omega t^2}.$$



   c. Observe that $\widehat{f} = \widehat{g}_\lambda \widehat{f}$. By substituting the expansion in part (a) into the Fourier inversion formula, show that

$$f(t) = \frac{1}{2\pi} \int_{-\lambda\Omega}^{\lambda\Omega} \widehat{f}(\omega)\widehat{g}_\lambda(\omega)e^{i\omega t}\, d\omega = \frac{\pi}{\lambda\Omega} \sum_{-\infty}^{\infty} f\left(\frac{n\pi}{\lambda\Omega}\right) g_\lambda\left(t - \frac{n\pi}{\lambda\Omega}\right).$$

   This gives a sampling formula for f in which the basic functions $g_\lambda(t)$ decay like t^{-2} at infinity.

9. Suppose that f satisfies the hypotheses of Heisenberg's inequality, and let $F(x) = e^{-i\alpha x} f(x + a)$.
   a. Show that $\Delta_a f = \Delta_0 F$.
   b. Show that $\widehat{F}(\xi) = e^{ia(\xi+\alpha)} \widehat{f}(\xi + \alpha)$ and thence that $\Delta_\alpha \widehat{f} = \Delta_0 \widehat{F}$.
10. Show that Heisenberg's inequality $(\Delta_0 f)(\Delta_0 \widehat{f}) \geq \frac{1}{4}$ is an equality if and only if $f' + cf = 0$ where c is a real constant, and hence show that the functions that minimize the uncertainty product $(\Delta_0 f)(\Delta_0 \widehat{f})$ are precisely those of the form $f(x) = Ce^{-cx^2/2}$ for some $c > 0$. (Hint: Examine the proof of Heisenberg's inequality and recall that the Cauchy-Schwarz inequality $|\int f\overline{g}| \leq \|f\|\,\|g\|$ is an equality if and only if f and g are scalar multiples of one another.) What are the minimizing functions for the uncertainty product $(\Delta_a f)(\Delta_\alpha \widehat{f})$ for general a, α? (Cf. Exercise 9.)

- Folland Section 3.3, problem 4: Suppose $\{\phi_n\}$ is an orthonormal basis for $L^2(a, b)$. Suppose $c > 0$ and $d \in \mathbb{R}$, and let $\psi_n(x) = c^{1/2}\phi_n(cx + d)$. Show that $\{\psi_n\}$ is an orthonormal basis for $L^2(\frac{a-d}{c}, \frac{b-d}{c})$.

- Folland Section 3.3, problem 11: Suppose f is of class $C^{(1)}$, 2π-periodic, and real-valued. Show that f' is orthogonal to f in $L^2(-\pi, \pi)$ in two ways: (a) by expanding f in a Fourier series and using Parseval's Theorem: $\langle f, g\rangle = \sum \langle f, \phi_n\rangle \overline{\langle g, \phi_n\rangle}$, (b) directly from the fact that $2ff' = (f^2)'$.

- Folland 2.6, problem 1 (Gibbs' phenomenon) (copied below). The referenced equations (2.10) and (2.12) are that the Nth Dirichlet kernel is given by

$$D_N(\phi) = \frac{1}{2\pi}\sum_{n=-N}^{n=N} e^{in\phi} = \frac{1}{2\pi}\frac{\sin((N+\frac{1}{2})\phi)}{\sin(\frac{1}{2}\phi)}$$

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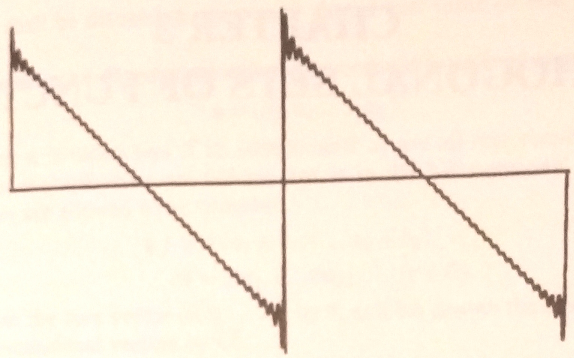

FIGURE 2.8. Graph of $2\sum_1^{40} n^{-1}\sin n\theta$, $-2\pi < \theta < 2\pi$ (an illustration of the Gibbs phenomenon).

EXERCISE

1. Recall from Table 1, §2.1, that $f(\theta) = 2\sum_1^\infty n^{-1}\sin n\theta$ is the 2π-periodic function that equals $\pi - \theta$ for $0 < \theta < 2\pi$. Let

$$g_N(\theta) = 2\sum_1^N \frac{\sin n\theta}{n} - (\pi - \theta),$$

so that $g(\theta)$ is the difference between $f(\theta)$ and its Nth partial sum for $0 < \theta < 2\pi$.

a. Show that $g_N'(\theta) = 2\pi D_N(\theta)$ where D_N is the Dirichlet kernel (2.10).

b. Using (2.12), show that the first critical point of $g_N(\theta)$ to the right of zero occurs at $\theta_N = \pi/(N+\frac{1}{2})$, and that

$$g_N(\theta_N) = \int_0^{\theta_N} \frac{\sin(N+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta}\,d\theta - \pi.$$

c. Show that

$$\lim_{N\to\infty} g_N(\theta_N) = 2\int_0^\pi \frac{\sin\phi}{\phi}\,d\phi - \pi.$$

(Hint: Let $\phi = (N+\frac{1}{2})\theta$.) This limit is approximately equal to .562. Thus the difference between $f(\theta)$ and the Nth partial sum of its Fourier series develops a spike of height .562 (but of increasingly narrow width) just to the right of $\theta = 0$ as $N \to \infty$. (There is another such spike on the left.)