M408D Final Exam Solutions, December 14, 2002

**Part I**

1. Evaluate the following limits and improper integrals, or write "DNE" if the limit does not exist. EXPLAIN YOUR REASONING.

$$\int_0^\infty \frac{2x}{x^2+1}dx$$

$\int_0^b \frac{2x}{x^2+1}dx = \ln(b^2+1)$, which diverges as $b \to \infty$, so the answer is DNE.

$$\lim_{x\to 3} \frac{\ln(x^2) - 2\ln(3)}{x^2-9}$$

Since $\ln(3^2) = 2\ln(2)$, this is a 0/0 indeterminate form. By L'Hôpital's rule, the limit is $(2x/x^2)/2x = 1/x^2 = 1/9$.

$$\lim_{x\to\infty} \frac{\sin(x^3)}{x^2}$$

The numerator is bounded while the denominator goes to $\infty$, so the limit is zero.

$$\lim_{n\to\infty} \ln(e^{\sqrt{n^2+7}}) - n.$$

Note that $\ln(e^{\sqrt{n^2+7}}) = \sqrt{n^2+7}$, and $\sqrt{n^2+7} - n = 7/(\sqrt{n^2+7}+n)$, which goes to zero.

2. Indicate which of these series converge absolutely, which converge conditionally, and which diverge. EXPLAIN YOUR REASONING.

a) $\displaystyle\sum_{n=2}^\infty \frac{(-1)^n}{\ln(n)}$

This is an alternating series that converges. However, $\sum 1/\ln(n)$ diverges by comparison to $\sum 1/n$, so the answer is CONVERGES CONDITIONALLY.

b) $\displaystyle\sum_{n=1}^\infty \frac{(-1)^n n^2 \ln(n)}{2^n}$

This CONVERGES ABSOLUTELY by the ratio test (or by the root test).

c) $\displaystyle\sum_{n=1}^\infty \frac{2n}{n^2+1}$

This DIVERGES by the integral test (compare to problem 1a).

d) $\displaystyle\sum_{n=1}^\infty \frac{5^n}{\sqrt{n!}}$

This CONVERGES ABSOLUTELY by the ratio (or root) test.

3. Consider the infinite power series $f(x) = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n^2}$.

a) Find the radius of convergence.

By the root test, the radius of convergence is $1/\lim((n^{-2})^{1/n}) = 1$.

b) Use the series to estimate $f(0.2)$. Your answer should be good to 3 decimal places.

$f(0.2) \approx 1 + 0.2 + (0.2)^2/4 + (0.2)^3/9 \approx 1 + 0.2 + 0.01 + 0.001 = 1.211$.

c) Estimate $\int_0^{0.1} f(x)dx$ to 3 decimal places.

Two terms are enough: $\int_0^{0.1}(1 + x + \cdots)dx = 0.1 + 0.1^2/2 + \cdots \approx 0.105$.

d) Estimate $f'(0.1)$ to 3 decimal places.

$f'(0.1) = 1 + (0.1)/2 + (0.1)^2/3 + \cdots \approx 1.053$

4. Let $L$ be the line through the points $(0,1,2)$ and $(3,2,3)$. Let $P$ be the plane containing the origin and the line $L$. Let $Q = (5,1,2)$.

a) Find the equation for the line $L$.

$d = (3,2,2) - (0,1,2) = (3,1,1)$, so our line is $\dfrac{x}{3} = \dfrac{y-1}{1} = \dfrac{z-2}{1}$.

b) Find the equation for the plane $P$.

We want the plane through (0,0,0), (0,1,2) and (3,2,3). The normal vector is $N = (0,1,2) \times (3,2,3) = (-1,6,-3)$, so the plane is $-x + 6y - 3z = 0$.

c) Find the distance from $Q$ to $L$.

$$\frac{\|(5,0,0) \times (3,1,1)\|}{\|(3,1,1)\|} = \frac{5\sqrt{2}}{\sqrt{11}}.$$

d) Find the distance from $Q$ to $P$.

$$\frac{|(5,0,0) \cdot (-1,6,-3)|}{\|(-1,6,-3)\|} = 5/\sqrt{46}.$$

5. Consider the parametrized curve $\mathbf{r}(t) = (1 + e^{-t}, \sqrt{2}t, e^t - 3)$

a) Find the velocity (vector) at time $t = 0$.

$v(t) = r'(t) = (-e^{-t}, \sqrt{2}, e^t)$, so $v(0) = (-1, \sqrt{2}, 1)$.

b) Find the speed as a function of time.

The speed is $\|v(t)\| = \sqrt{e^{-2t} + 2 + e^{2t}} = e^t + e^{-t}$.

c) Find the distance traveled (i.e., arclength) from $t = -1$ to $t = 1$.

Distance $= \int$ speed $dt = \int_{-1}^{1}(e^t + e^{-t})dt = 2(e - e^{-1})$.

## Part II

6. a) The width of a rectangle is increasing at a rate of 0.2 meters/second, while the height is increasing at a rate of 1 meter/second. At what rate is the area increasing when the height is 10 meters and the width is 0.5 meters?

$A = wh$, where A is area, w is width and h is height. $dA/dt = \partial A/\partial w(dw/dt)+\partial A/\partial h(dh/dt) = h(dw/dt)+w(dh/dt) = 10(0.2)+0.5(1) =$ 2.5 square meters per second.

b) A bird is flying through a forest fire at 6 meters/second, in the direction (2,2,1). The temperature as a function of position is $f(x, y, z) = xy + 2z^2$. At what rate is the temperature changing when the bird passes the point (5,20,10)?

Velocity is a multiple of (2,2,1), so $v = k(2, 2, 1)$ for some constant $k$. Since $6 = \|v\| = k\sqrt{2^2 + 2^2 + 1} = 3k$, we have $k = 2$, so $v = (4, 4, 2)$.

$$\frac{df}{dt} = \nabla f \cdot v = (y, x, 4z) \cdot (4, 4, 2) = 4y + 4x + 8z = 180.$$

7. Consider the surface $x^2y + yz + z^4 = 3$.

a) Find the equation of the plane tangent to this surface at the point $(1, 1, 1)$.

The normal vector is $\nabla f = (2xy, x^2 + z, y + 4z^3) = (2, 2, 5)$, so our plane is $2x + 2y + 5z = 9$.

b) Find the equation of the line normal to this surface at the point $(1, 1, 1)$.

Since $d = (2, 2, 5)$, our line is $\frac{x-1}{2} = \frac{y-1}{2} = \frac{z-1}{5}$.

8. Consider the function of two variables: $f(x, y) = (3 + e^x - x)(e^{y^2} + 5)$.

a) Find all critical points of this function.

$\partial f/\partial x = (e^x - 1)(e^{y^2} + 5)$ and $\partial f/\partial y = 2ye^{y^2}(3 + e^x - x)$. Setting these equal to zero gives $x = y = 0$. This is the only critical point.

b) Use the second derivative test to determine which critical points are local maxima, which are local minima, and which are saddle points.

$A = \partial^2/\partial x^2 = e^x(e^{y^2} + 5) = 6$, $B = 2ye^{y^2}(e^x - 1) = 0$, and $C = (2 + 4y^2)e^{y^2}(3 + e^x - x) = 8$, so we are at a local MINIMUM.

9. Find the maximum and minimum values of the function $f(x, y, z) = x^2 + 4yz$ on the unit sphere $x^2 + y^2 + z^2 = 1$. Where do these maximum and minimum values occur?

$\nabla f = (2x, 4z, 4y)$ and $\nabla g = (2x, 2y, 2z)$, so our equations are

$$2x = \lambda(2x), \qquad 4z = \lambda(2y), \qquad (4y = \lambda(2z), \qquad x^2 + y^2 + z^2 = 1.$$

The solutions are $(x, y, z, \lambda) = (1, 0, 0, 1)$, $(-1, 0, 0, 1)$, $(0, \sqrt{2}/2, \sqrt{2}/2, 2)$, $(0, -\sqrt{2}/2, -\sqrt{2}/2, 2)$, $(0, \sqrt{2}/2, -\sqrt{2}/2, -2)$, and $(0, -\sqrt{2}/2, \sqrt{2}/2, -2)$. The first two have $f = 1$, the next two have $f = 2$, and the last two have $f = -2$, so the last two are minima, the middle two are maxima, and the first two are neither. That is, the maximum value is 2, and is achieved at $\pm(0, \sqrt{2}/2, \sqrt{2}/2)$, while the minimum value is 2, and is achieved at $\pm(0, \sqrt{2}/2, -\sqrt{2}/2)$.

10. Let $T$ be the triangle in the $x$-$y$ plane with vertices $(0, 0)$, $(2, 0)$ and $(1, 1)$. We are interested in the double integral $\int\int_T 6xydA$.

a) Write $\int\int_T 6xydA$ as an iterated integral, where you integrate first over $x$ and then over $y$. (Do not evaluate yet)

$$\int_{y=0}^{1} \int_{x=y}^{2-y} 6xydxdy.$$

b) Break $T$ up into two smaller triangles, $T_1$ with vertices at $(0, 0)$, $(1, 0)$ and $(1, 1)$, and $T_2$ with vertices at $(1, 0)$, $(2, 0)$ and $(1, 1)$. Write $\int\int_{T_1} 6xydA$ as an iterated integral, where you integrate first over $y$ and then over $x$. Do the same for $\int\int_{T_2} 6xydA$.

$$\int_{x=0}^{1} \int_{y=0}^{x} 6xydydx \quad \text{and} \quad \int_{x=1}^{2} \int_{y=0}^{2-x} 6xydydx.$$

c) Compute $\int\int_T 6xydA$, either by doing the iterated integral you wrote down in (a) or the ones you wrote down in (b).

In (a), $\int_y^{2-y} 6xydx = 3x^2y|_{x=y}^{x=2-y} = 12y - 12y^2$, so our outer integral is $\int_0^1 (12y - 12y^2)dy = 6y^2 - 4y^3|_0^1 = 2$.