M408D First Midterm Exam Solutions, October 3, 2002

1. A sequence $\{a_n\}$ is said to "grow faster" than $\{b_n\}$ if $\lim_{n\to\infty} b_n/a_n = 0$. Put the following sequences in order of growth rate, from fastest to slowest. Justify your answers.

$a_n = \sqrt{3^n + 5^n}$,
$b_n = \ln(n^{1000})$,
$c_n = \sqrt{n}$,
$d_n = (n!)^{1/100}$,
$e_n = \ln(5e^n)$.

Note that $\ln(5e^n) = n + \ln(5)$ and that $\ln(n^{1000}) = 1000\ln(n)$. The sequence d_n (factorial) grows faster than a_n (exponential), which grows faster than e_n (power), which grows faster than c_n (smaller power), which grows faster than b_n (log).

2. Evaluate the following limits or improper integrals, or write DNE if the limits do not exist.

a) $\lim\limits_{x\to 0}\left(1+\frac{x}{3}\right)^{1/x}$. If you let $n = 1/x$, this is $\lim(1+(1/3)/n)^n = e^{1/3}$.

b) $\int_0^\infty xe^{-x}dx$. Since $\int xe^{-x}dx = -(1+x)e^{-x}$ (integrate by parts), we have $\int_0^b xe^{-x}dx = 1-(1+b)e^{-b}$, which goes to 1 as $b \to \infty$.

c) $\lim\limits_{x\to 0}\frac{1-\cos(x)}{\sin^2(x)}$. This is a "0/0" indeterminate form. Applying L'Hopital's rule once gives $\lim\limits_{x\to 0}\frac{\sin(x)}{2\sin(x)\cos(2)}$, which equals 1/2.

d) $\lim\limits_{x\to 5}\frac{\sqrt{x^2-25}}{x^2-6x+4}$. This is NOT an indeterminate form. It is $0/(-1) = 0$.

3. Which of the following series and integrals converge, and which diverge. In each case, give a 1-sentence explanation (e.g., "converges by ratio test", or "diverges by comparison to 2^k")

a) $\sum \frac{k}{k^2+5}$ diverges by integral test, or by comparison to $1/2k$.

b) $\sum \frac{k^{15}}{2^k}$ converges by ratio test or by root test.

c) $\sum k!e^{-k}$ diverges. Terms don't go to zero.

d) $\sum \frac{\cos(\pi k)}{k}$. This is a funny way of writing the alternating series $\sum(-1)^k/k$, which converges.

4. a) Write down the Taylor series for e^x.

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + x^2/2 + x^3/6 + \cdots.$$

b) Write down the 6-th order Taylor polynomial (i.e., through the x^6 term) for $f(x) = e^{-4x^3}$.

$f(x) = e^{-4x^3} = 1 + (-4x^3) + (-4x^3)^2/2 + \cdots = 1 - 4x^3 + 8x^6 + \cdots$, so the answer is $1 - 4x^3 + 8x^6$.

c) Evaluate $f^{(6)}(0)$. (That is, the 6-th derivative of $f(x)$, evaluated at $x = 0$.) The coefficient of x^6, namely 8, is the answer divided by 6!, so the answer must be $8(6!) = 8(720) = 5760$.

d) Evaluate $\int_0^{0.1} f(x)dx$ to five decimal places. (No, you don't need a calculator!)

The third-order Taylor polynomial is good enough:
$\int_0^{0.1} f(x)dx \approx \int_0^{0.1} 1 - 4x^3 dx = x - x^4|_0^{0.1} = 0.09990.$

5. a) Find the second-order Taylor polynomial for $f(x) = \sqrt{x}$ around $x = 4$.

$f(x) = x^{1/2}$, $f'(x) = (1/2)x^{-1/2}$ and $f''(x) = -(1/4)x^{-3/2}$. Taking $a = 4$ we have $f(a) = 2$, $f'(a) = 1/4$ and $f''(a) = -1/32$, so

$$f_2(x) = 2 + (1/4)(x - 2) - (1/64)(x - 2)^2.$$

b) What is the radius of convergence of the power series $\sum_{k=0}^{\infty} \frac{2k^2+17}{2^k} x^k$? Justify your answer.

By the root test, $\lim |a_n^{1/n}| = 1/2$, so the radius of convergence is 2. (The ratio test would work just as well).