M346 Final Exam
December 13, 2000

Problem 1.

Consider the vector space $M_{2,2}$ of 2×2 matrices, let $B = \begin{pmatrix} 0 & 2 \\ 3 & 5 \end{pmatrix}$. Consider the linear transformations $L_1(A) = AB$ and $L_2(A) = BA$.
a) Find the matrix of L_1 relative to the basis

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

b) Find the matrix of L_2 relative to the same basis.

Problem 2. Let $A = \frac{1}{4}\begin{pmatrix} -4 & 3 \\ 7 & 0 \end{pmatrix}$.

a) Consider the equations $\mathbf{x}(n) = A\mathbf{x}(n-1)$, with A as above. What are the stable and unstable modes? What is the dominant eigenvector?

b) Consider the equations $\dot{\mathbf{x}}(t) = A\mathbf{x}(t)$, with A as above. What are the stable and unstable modes? What is the dominant eigenvector?

Problem 3. Let $A = \frac{1}{5}\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}$. Which of the following are Hermitian? Which are unitary? Which are both? Which are neither?

a) A
b) $A + I$
c) e^A
d) e^{iA}

Problem 4. In R^4 with the standard inner product, consider the vectors $\mathbf{b}_1 = (1,0,0,1)^T$, $\mathbf{b}_2 = (1,2,2,1)^T$, $\mathbf{b}_3 = (2,1,1,0)^T$, $\mathbf{b}_4 = (1,3,5,7)^T$. Apply Gram-Schmidt to turn this into an orthogonal basis for $\mathbf{R}^4$.

Problem 5. Consider a sequence of numbers satisfying the second order difference equation $x(n) = 2x(n-1) + 3x(n-2)$ for $n \geq 2$.

a) Reduce this 2nd order difference equation to a 2×2 system of first order difference equations.

b) Find the most general solution to the first order system.

c) From initial data $x(0) = 2$, $x(1) = 2$, find $x(n)$ for all n.

Problem 6. Consider the nonlinear system of equations

$$\begin{aligned} x_1(n) &= 1 - x_1(n-1)x_2(n-1), \\ x_2(n) &= x_1(n-1)^2 + x_2(n-1)^2 - 1. \end{aligned}$$

a) Linearize this system of equations near the fixed point $(1,0)^T$.
b) Find the modes and determine which are stable and which are unstable.
c) Is the fixed point $(1,0)^T$ stable?

Problem 7. Diagonalize the matrix $A = \begin{pmatrix} 2 & 3 & 1 & 4 \\ -3 & 2 & 4 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 2 & 1 \end{pmatrix}$

Problem 8.
We wish to solve the differential equation

$$\frac{\partial^2 f(x,t)}{\partial t^2} = \frac{\partial^2 f(x,t)}{\partial x^2} - f(x,t) \qquad (KG)$$

on the interval $(0,\pi)$ with Dirichlet boundary conditions:

$$f(0,t) = f(\pi,t) = 0$$

for all t. [This is called the Klein-Gordon equation, and comes up in relativistic quantum mechanics. We have not studied this equation, but you can solve it using the same ideas that gave us standing waves solutions to the wave equation.]
a) Find the eigenvalues and eigenfunctions of the operator $d^2/dx^2 - 1$ (with Dirichlet boundary conditions).
b) Find the most general solution to (KG).
c) Given the initial conditions $f(x,0) = \sin(2x)$, $\dot{f}(x,0) = \sin(4x)$, find $f(x,t)$ for all $x \in (0,\pi)$ and all t.