M346 Second Midterm Exam, October 23, 2003

1. Find all the eigenvalues of the following matrices. You do NOT need to find the corresponding eigenvectors. [Note: the answers are fairly simple, and can be obtained without a lot of calculation, using the various "tricks of the trade".]

a) $\begin{pmatrix} 3 & 1 & 5 & 17 \\ 1 & 3 & 4 & -10 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 0 & 2 \\ 1 & 2 & 3 \end{pmatrix}$.

2. The eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}$ are $\lambda_1 = -2$, $\lambda_2 = 1$ and $\lambda_3 = 1$, and corresponding eigenvectors $\mathbf{b}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{b}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $\mathbf{b}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$. (That is, the eigenvalue 1 has multiplicity two, and a basis for the eigenspace E_1 is $\{\mathbf{b}_2, \mathbf{b}_3\}$.)

a) Solve the difference equation $\mathbf{x}(n+1) = A\mathbf{x}(n)$ with initial condition $\mathbf{x}(0) = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$ (which equals $\mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3$, by the way). That is, find $\mathbf{x}(n)$ for every n.

b) With the situation of part (a), identify the stable, unstable, and neutrally stable modes. What are the limiting ratios $x_1(n)/x_2(n)$ and $x_1(n)/x_3(n)$ when n is large?

c) Now solve the differential equation $d\mathbf{x}/dt = A\mathbf{x}$ with initial condition $\mathbf{x}(0) = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$. That is, find $\mathbf{x}(t)$ for all t.

d) With the situation of part (c), identify the stable, unstable, and neutrally stable modes. What are the limiting ratios $x_1(t)/x_2(t)$ and $x_1(t)/x_3(t)$ when t is large?

3. Consider the matrix $A = \begin{pmatrix} 4 & 5 \\ 5 & 4 \end{pmatrix}$.

a) Find the eigenvalues and eigenvectors of A.

b) Write down the general solution to the second-order differential equation $d^2\mathbf{x}/dt^2 = A\mathbf{x}$, with A as above.

c) Find the solution to this equation when $\mathbf{x}(0) = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$ and $\dot{\mathbf{x}}(0) = \begin{pmatrix} 11 \\ 1 \end{pmatrix}$.

4. A 2×2 matrix M has eigenvalues 1 and 8, and corresponding eigenvectors $\mathbf{b}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\mathbf{b}_2 = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$. Consider the basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ for $\mathbb{R}^2$.

a) Find $[M]_{\mathcal{B}}$, $P_{\mathcal{E}\mathcal{B}}$ and $P_{\mathcal{B}\mathcal{E}}$.

b) Find M (expressed in the ordinary basis).

c) A matrix A has the property that $A^3 = M$. Find A. [Hint: what are the eigenvalues and eigenvectors of A?]

5. True of False? Each question is worth 4 points. You do NOT need to justify your answers, and partial credit will NOT be given.

a) The geometric multiplicity of an eigenvalue λ is the dimension of the eigenspace E_λ.

b) If a matrix is diagonalizable, then its eigenvalues are all different.

c) Let A by an arbitrary $n \times n$ matrix. The sum of the algebraic multiplicities of the eigenvalues of A must equal n.

d) The eigenvalues of a (square) matrix with real entries are always real.

e) If $B = PAP^{-1}$, then A and B have the same eigenvalues.