M346 Final Exam, December 10, 2003

1. In the vector space $\mathbb{R}_2[t]$, consider the basis $\mathcal{B} = \{1, 1+t, 1+t+t^2\}$, the basis $\mathcal{D} = \{1+t+t^2, t+t^2, t^2\}$ and the vector $\mathbf{v} = 2t^2 - 1$.

a) Find $[\mathbf{v}]_{\mathcal{B}}$ and $[\mathbf{v}]_{\mathcal{D}}$. (That is, find the coordinates of $\mathbf{v}$ in the $\mathcal{B}$ basis, and the coordinates of $\mathbf{v}$ in the $\mathcal{D}$ basis.)

b) Find the change-of-basis matrices $P_{\mathcal{B}\mathcal{D}}$ and $P_{\mathcal{D}\mathcal{B}}$.

2. Consider the operator $L : \mathbb{R}_2[t] \to \mathbb{R}_2[t]$ defined by $(L\mathbf{p})(t) = \mathbf{p}(t+1) - \mathbf{p}(t)$.

a) Find the matrix of L in the standard basis $\{1, t, t^2\}$.

b) Find the matrix of L in the basis $\mathcal{B} = \{1, 1+t, 1+t+t^2\}$ (this is the same basis $\mathcal{B}$ you saw in problem 1).

3. Matrices and eigenvalues:

a) Find the eigenvalues and eigenvectors of the matrix $\begin{pmatrix} 2 & 1 & 4 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$.

b) Find a matrix whose eigenvalues are $1, 3, 4$ and eigenvectors are $\frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$, $\frac{1}{3}\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$ and $\frac{1}{3}\begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix}$. [Hint: there is an easy way to compute P^{-1}]

4. Let $A = \begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix}$. (You may find useful the fact that A is Hermitian, but you don't need this fact to solve the problem).

a) Find the eigenvalues and eigenvectors of the matrix A.

b) Find the most general solution to the system of equations $\frac{d^2\mathbf{x}}{dt^2} = A\mathbf{x}$.

c) Find the solution to the system of equations $\frac{d^2\mathbf{x}}{dt^2} = A\mathbf{x}$ with initial conditions $\mathbf{x}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, $\dot{\mathbf{x}}(0) = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$.

5. Linearization in one dimension.

a) Consider the first order differential equation $dx/dt = f(x)$, where $f(x) = \frac{1}{9}\sin(\pi x^2)$. This has fixed points at $x = 1$, $x = 2$, and $x = 3$ (and lots of other points, which we'll ignore). Which of these three points is stable, unstable, or neutrally stable?

b) Next, consider the second-order differential equation $d^2x/dt^2 = f(x)$, where $f(x) = \frac{1}{9}\sin(\pi x^2)$, as before. Again, this has fixed points at $x = 1, 2, 3$. Which of these three points is stable, unstable, or neutrally stable?

c) Finally, consider the difference equation $x(n+1) = g(x(n))$, where now $g(x) = x + \frac{1}{9}\sin(\pi x^2)$. Yet again, this has fixed points at $x = 1, 2, 3$. Which of these three points is stable, unstable, or neutrally stable?

6. Consider the vectors $\mathbf{b}_1 = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$, $\mathbf{b}_2 = \begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix}$, $\mathbf{b}_3 = \begin{pmatrix} 6 \\ 4 \\ -9 \end{pmatrix}$ in $\mathbb{R}^3$. Note that these vectors are orthogonal.

a) Decompose the vector $\mathbf{v} = \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix}$ as a linear combination of $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$. [Warning: this problem involves somewhat messy fractions.]

b) Find the matrices for the projections $P_{\mathbf{b}_1}$ and $P_{\mathbf{b}_2}$.

c) Find an orthogonal matrix whose columns are proportional to the three vectors $\mathbf{b}_i$.

7. Working on the interval $x \in [0, 1]$, let

$$f(x) = \begin{cases} 1 & \text{if } 1/4 < x < 3/4; \\ 0 & \text{otherwise} \end{cases}; \qquad g(x) = 2x - 1.$$

The functions $f(x)$ and $g(x)$ can each be written as (sine) Fouries series:

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x); \qquad g(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x).$$

a) Compute a_1, a_2, a_3 and a_4.

b) Compute $\sum_{n=1}^{\infty} |a_n|^2$.

c) Compute $\sum_{n=1}^{\infty} a_n b_n$. [Note: you do NOT need the results of (a) to do (b) and (c)]

8. We wish to solve the differential equation

$$\frac{\partial f(x,t)}{\partial t} = \frac{\partial^2 f(x,t)}{\partial x^2}$$

on the interval $x \in (0, \pi)$ with Dirichlet boundary conditions $f(0,t) = f(\pi, t) = 0$. [This is called the heat equation, and can be attacked by generalizing the methods of section 5.1 to spaces of functions, just as the vibrating string problem was solved by generalizing the methods of section 5.3 to functions.]

(a) Find the most general solution to this equation.

b) Given the initial conditions $f(x,0) = 3\sin(x) - 5\sin(2x) + 37\sin(3x)$, find $f(x,t)$ for all $x \in (0,\pi)$ and all t. [Note that the equation involves the first derivative with respect to time, so our initial conditions are just the value $f(x,0)$ of the function at time $t = 0$, and doesn't involve $\dot{f}(x,0)$.]

9. True of False? Each question is worth 2 points. You do NOT need to justify your answers, and partial credit will NOT be given.

a) If a matrix is Hermitian, then the geometric multiplicity of each eigenvalue equals the algebraic multiplicity.

b) If A is a real anti-symmetric matrix ($A^T = -A$), then e^A is an orthogonal matrix.

c) Every solution to the wave equation on the real line is either a forward traveling wave or a backwards traveling wave.

d) There exists a Hermitian matrix with eigenvalue $2 + i$.

e) The equation $A\mathbf{x} = \mathbf{b}$ has a least-squares solution only if $\mathbf{b}$ is in the column space of A.

f) Every change-of-basis matrix is invertible.

g) If A is a 4×7 matrix, then the null space of A is 3-dimensional.

h) If the columns of a square matrix are linearly dependent, then zero is an eigenvalue.

i) The system $d\mathbf{x}/dt = A\mathbf{x}$ is stable if all the eigenvalues of A lie inside the unit circle.

j) If $\mathcal{B}, \mathcal{D}$ and $\mathcal{E}$ are bases for the same vector space, then the change-of-basis matrices satisfy $P_{\mathcal{B}\mathcal{D}} = P_{\mathcal{B}\mathcal{E}} P_{\mathcal{E}\mathcal{D}}$.