M346 Final Exam Soutions
December 13, 2000

Problem 1.

Consider the vector space $M_{2,2}$ of 2×2 matrices, let $B = \begin{pmatrix} 0 & 2 \\ 3 & 5 \end{pmatrix}$. Consider the linear transformations $L_1(A) = AB$ and $L_2(A) = BA$.

a) Find the matrix of L_1 relative to the basis

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

We compute:

$$\begin{aligned}
L_1\mathbf{b}_1 &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = 2\mathbf{b}_2 \\
L_1\mathbf{b}_2 &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 0 & 0 \end{pmatrix} = 3\mathbf{b}_1 + 5\mathbf{b}_2 \\
L_1\mathbf{b}_3 &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} = 2\mathbf{b}_4 \\
L_1\mathbf{b}_4 &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 3 & 5 \end{pmatrix} = 3\mathbf{b}_3 + 5\mathbf{b}_4
\end{aligned}$$

so we have

$$[L_1]_{\mathcal{B}} = \begin{pmatrix} 0 & 3 & 0 & 0 \\ 2 & 5 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 2 & 5 \end{pmatrix}$$

b) Find the matrix of L_2 relative to the same basis.

We compute:

$$\begin{aligned}
L_2\mathbf{b}_1 &= \begin{pmatrix} 0 & 2 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix} = 3\mathbf{b}_3 \\
L_1\mathbf{b}_2 &= \begin{pmatrix} 0 & 2 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix} = 3\mathbf{b}_4 \\
L_1\mathbf{b}_3 &= \begin{pmatrix} 0 & 2 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 5 & 0 \end{pmatrix} = 2\mathbf{b}_1 + 5\mathbf{b}_3 \\
L_1\mathbf{b}_4 &= \begin{pmatrix} 0 & 2 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 5 \end{pmatrix} = 2\mathbf{b}_2 + 5\mathbf{b}_4
\end{aligned}$$

so we have

$$[L_2]_{\mathcal{B}} = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 3 & 0 & 5 & 0 \\ 0 & 3 & 0 & 5 \end{pmatrix}$$

Problem 2. Let $A = \frac{1}{4}\begin{pmatrix} -4 & 3 \\ 7 & 0 \end{pmatrix}$.

a) Consider the equations $\mathbf{x}(n) = A\mathbf{x}(n-1)$, with A as above. What are the stable and unstable modes? What is the dominant eigenvector?

The eigenvalues are $\lambda_1 = 3/4$ and $\lambda_2 = -7/4$, with eigenvectors $\mathbf{b}_1 = (3,7)^T$ and $\mathbf{b}_2 = (1,-1)^T$. The first mode is stable since $|\lambda_1| < 1$, while the second is unstable since $|\lambda_2| > 1$. The dominant eigenvector is $\lambda_2 = -7/4$.

b) Consider the equations $\dot{\mathbf{x}}(t) = A\mathbf{x}(t)$, with A as above. What are the stable and unstable modes? What is the dominant eigenvector?

Now the question is whether the real part of λ is positive or negative. Since $\lambda_1 > 0$, the first mode is unstable. Since $\lambda_2 < 0$, the second mode is stable. Now the dominant eigenvector is λ_1.

Problem 3. Let $A = \frac{1}{5}\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}$. Which of the following are Hermitian? Which are unitary? Which are both? Which are neither?

Notice that the eigenvalues of A are ± 1, and the eigenvectors are orthogonal. You can see this by calculating them [they are $(2,1)^T$ and $(1,-2)^T$], or from the fact that A is manifestly Hermitian. The various operations all give matrices with the same eigenvectors as A, but different eigenvalues. Since the eigenvectors are orthogonal, a matrix will be Hermitian if its eigenvalues are real, and unitary if its eigenvalues have norm one.

a) A

Both Hermitian and unitary, since 1 and -1 are both real and of norm 1.

b) $A + I$

Hermitian but not unitary, since 2 and 0 are real but not of norm 1.

c) e^A

Hermitian but not unitary, since $e^{\pm 1}$ are real but not of norm 1.

d) e^{iA}

Unitary but not Hermitian, since $e^{\pm i}$ are complex but of norm 1.

Problem 4. In R^4 with the standard inner product, consider the vectors $\mathbf{b}_1 = (1,0,0,1)^T$, $\mathbf{b}_2 = (1,2,2,1)^T$, $\mathbf{b}_3 = (2,1,1,0)^T$, $\mathbf{b}_4 = (1,3,5,7)^T$. Apply Gram-Schmidt to turn this into an orthogonal basis for $\mathbf{R}^4$.

$y_1 = \mathbf{b}_1 = (1,0,0,1)^T$.

$y_2 = \mathbf{b}_2 - \frac{\langle y_1|\mathbf{b}_2\rangle}{\langle y_1|\mathbf{y}_1\rangle} y_1 = (1,2,2,1)^T - (1,0,0,1)^T = (0,2,2,0)^T$.

$y_3 = \mathbf{b}_3 - \frac{\langle y_1|\mathbf{b}_3\rangle}{\langle y_1|\mathbf{y}_1\rangle} y_1 - \frac{\langle y_2|\mathbf{b}_3\rangle}{\langle y_2|\mathbf{y}_2\rangle} y_2 = (2,1,1,0)^T - (2/2)(1,0,0,1)^T - (4/8)(0,2,2,0)^T = (1,0,0,-1)^T$.

$y_4 = \mathbf{b}_4 - \frac{\langle y_1|\mathbf{b}_4\rangle}{\langle y_1|\mathbf{y}_1\rangle} y_1 - \frac{\langle y_2|\mathbf{b}_4\rangle}{\langle y_2|\mathbf{y}_2\rangle} y_2 - \frac{\langle y_3|\mathbf{b}_4\rangle}{\langle y_3|\mathbf{y}_3\rangle} y_3 = (1,3,5,7)^T - (8/2)(1,0,0,1)^T - (16/8)(0,2,2,0)^T - (-6/2)(1,0,0,-1)^T = (0,-1,1,0)^T$.

Problem 5. Consider a sequence of numbers satisfying the second order difference equation $x(n) = 2x(n-1) + 3x(n-2)$ for $n \geq 2$.

a) Reduce this 2nd order difference equation to a 2×2 system of first order difference equations.

Let $y(n) = x(n-1)$. Then $x(n) = 2x(n-1) + 3y(n-1)$, and we have

$$\begin{pmatrix} x(n) \\ y(n) \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x(n-1) \\ y(n-1) \end{pmatrix}.$$

b) Find the most general solution to the first order system.

The eigenvalues of the matrix are 3 and -1, with eigenvectors $(3,1)^T$ and $(1,-1)^T$, so the most general solution is

$$\begin{pmatrix} x(n) \\ y(n) \end{pmatrix} = c_1 3^n \begin{pmatrix} 3 \\ 1 \end{pmatrix} + c_2(-1)^n \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

c) From initial data $x(0) = 2$, $x(1) = 2$, find $x(n)$ for all n.

We have $\begin{pmatrix} x(1) \\ y(1) \end{pmatrix} = (2,2)^T = \mathbf{b}_1 - \mathbf{b}_2$, so $c_1 = 1/3$ and $c_2 = 1$. Thus $x(n) = 3^n + (-1)^n$.

Problem 6. Consider the nonlinear system of equations

$$\begin{aligned} x_1(n) &= 1 - x_1(n-1)x_2(n-1), \\ x_2(n) &= x_1(n-1)^2 + x_2(n-1)^2 - 1. \end{aligned}$$

a) Linearize this system of equations near the fixed point $(1,0)^T$.

$$A = \begin{pmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{pmatrix} = \begin{pmatrix} -x_2 & x_1 \\ 2x_1 & 2x_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}$$

Defining $\mathbf{y} = \mathbf{x} - (1,0)^T$, our linearized equations are $\mathbf{y}(n) \approx A\mathbf{y}(n-1)$.

b) Find the modes and determine which are stable and which are unstable.

The eigenvalues of A are $\pm i\sqrt{2}$, with eigenvectors $(1, \mp i\sqrt{2})^T$. Since both eigenvalues are (in magnitude) bigger than 1, both modes are unstable.

c) Is the fixed point $(1,0)^T$ stable?

And so $(1,0)^T$ is an unstable fixed point.

Problem 7. Diagonalize the matrix $A = \begin{pmatrix} 2 & 3 & 1 & 4 \\ -3 & 2 & 4 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 2 & 1 \end{pmatrix}$

Since it is block triangular, to find the eigenvalues you just need to diagonalize each block. The eigenvalues of the upper left block are $2 \pm 3i$, while the eigenvalues of the lower right block are 4 and 1. The eigenvectors are $\mathbf{b}_1 = (1, i, 0, 0)^T$, $\mathbf{b}_2 = (1, -i, 0, 0)^T$, $\mathbf{b}_3 = (4.9226, -0.3846, 3, 2)^T$, $\mathbf{b}_4 = (1, 0, 1, -2^T)$. Computing the first two entries of $\mathbf{b}_3$ is messy. I'll accept an answer of (junk, junk, 3,2)^T.

Problem 8.

We wish to solve the differential equation

$$\frac{\partial^2 f(x,t)}{\partial t^2} = \frac{\partial^2 f(x,t)}{\partial x^2} - f(x,t) \qquad (KG)$$

on the interval $(0, \pi)$ with Dirichlet boundary conditions:

$$f(0,t) = f(\pi, t) = 0$$

for all t. [This is called the Klein-Gordon equation, and comes up in relativistic quantum mechanics. We have not studied this equation, but you can solve it using the same ideas that gave us standing waves solutions to the wave equation.]

a) Find the eigenvalues and eigenfunctions of the operator $d^2/dx^2 - 1$ (with Dirichlet boundary conditions).

We already know the eigenvalues and eigenvectors of d^2/dx^2, namely $-n^2\pi^2/L^2 = -n^2$ and $\sin(n\pi x/L) = \sin(nx)$. The eigenvalues of $d^2/dx^2 - 1$ are just one less ($\lambda_n = -(n^2+1)$) and the eigenvectors are the same.

b) Find the most general solution to (KG).

Let $\omega_n = \sqrt{-\lambda_n} = \sqrt{n^2+1}$. Then

$$f(x,t) = \sum_{n=1}^{\infty} (a_n \cos(omega_n t) + b_n \sin(\omega_n t)) \sin(nx).$$

c) Given the initial conditions $f(x, 0) = \sin(2x)$, $\dot f(x, 0) = \sin(4x)$, find $f(x,t)$ for all $x \in (0, \pi)$ and all t.

The only nonzero coefficients are $a_2 = 1$ and $b_4 = 1/\sqrt{17}$, so

$$f(x,t) = \cos(\sqrt{5}t)\sin(2x) + \frac{\sin(\sqrt{17}t)}{\sqrt{17}}\sin(4x)$$