

Review of Chapter 16: Multiple Integrals

Note: This review sheet is NOT meant to be a comprehensive overview of what you need to know for the exam. It is merely another tool to help you get started studying. The following concepts may or may not be seen on the exam and there may be concepts on the exam which are not covered on this sheet.

- If you are rusty on INTEGRATION TECHNIQUES, take some time to review chapters 5 and 8 of your text. In particular, be sure you are comfortable using u -substitution (page 360) and integration by parts (page 511), as these are two of the most frequently used methods of integration.

Double Integrals

- EVALUATING A DOUBLE INTEGRAL OVER A GENERAL REGION: As a first step, sketch the 2-D region that you are integrating over, and decide whether it is a Type I or Type II region. In a Type I region, the y values are bounded between two continuous functions of x . In a Type II region, the x values are bounded between two continuous functions of y .

EXERCISE: Compare the two types of regions and their corresponding integrals, set up with the appropriate bounds, on pages 1032 and 1033 of your text. Note that the bounds on the “inside” integral are functions and the bounds on the “outside” integral are constants. Note how the type of region you are integrating over determines the order of integration.

- FUBINI’S THEOREM told us that when integrating over a rectangle,

$$\int \int_R f(x, y) dA = \int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy$$

Here we can easily switch the order of integration since a rectangle can obviously be viewed as a Type I or a Type II region. We can usually switch the order of integration over general regions as well, although we need to change our bounds accordingly. As we saw in examples during discussion section, one order of integration may be more difficult to work with than the other, depending on the type of region.

QUESTION: How can you express the **area** of a region, D , using a double integral?

- DOUBLE INTEGRALS IN POLAR COORDINATES: Sometimes it is easier to evaluate a double integral using polar coordinates. You should suspect that this is the case if the region you are integrating over is easily expressed using polar coordinates (something like a circle or an annulus), and when the integrand (the function which you are integrating) is easily expressed using polar coordinates.

EXERCISE: What are some examples of functions that you’ve seen in homework which are easier to integrate using polar coordinates? Could you pick these functions out of a crowd?

- When CONVERTING a double integral expressed in cartesian coordinates, like $\int \int_D f(x, y) dA$, to one using polar coordinates, these steps may be useful to follow:
 - (1) Sketch the region over which you are integrating and express that region using polar coordinates. This should help you set up the bounds for your integral in polar coordinates.
 - (2) Rewrite your function, $f(x, y)$, using polar coordinates, i.e. $f(r \cos \theta, r \sin \theta)$.
 - (3) Convert the term dA to polar coordinates, making sure not to lose the extra factor of r : $dA = r dr d\theta$.

* Remember that (just like with double integrals in cartesian coordinates) the inside bounds of your double integral in polar coordinates may involve functions. The simplest region to integrate over in cartesian coordinates is a rectangle: then both sets of bounds are just constants, because the equations of the lines bounding a rectangle in the xy plane are $x = a$ and $x = b$; $y = c$ and $y = d$. The simplest region to integrate over in polar coordinates is a circle or annulus centered at the origin. This is because the equation of a circle centered at the origin is just $r = a$ for some constant a , making the bounds on your double integral simply constants. However, if you have a more complicated region to integrate over using polar coordinates, say for instance the region $r \leq 2 \cos \theta$ (a disc of radius 1, shifted to the right by one unit), then the bounds on the double integral must involve this function of r . See for example problem 32 in section 16.4.

- There are many different applications of double integrals in fields such as physics, statistics, and economics. Some of these applications are covered in sections 16.5 and 16.6. These problems are useful to work through because they present you with double integrals in a new setting, so you have to make problem solving decisions, such as whether to use cartesian or polar coordinates.

Triple Integrals

- ORDER OF INTEGRATION: When evaluating double integrals, you had to choose whether to integrate with respect to x or with respect to y first. How many different orders of integration are there to choose from when evaluating the triple integral, $\int \int \int_B f(x, y, z) dV$?
- *How will you decide which variable (x, y, or z) to integrate with respect to first?* When working with double integrals, we wanted to integrate first over the variable that we could bound between two continuous functions of the other variable. The situation is analogous for triple integrals: We integrate first over a variable which we can bound between two continuous functions of the other two variables, i.e. between two nice surfaces.

*Remember here that if we are integrating with respect to x first, for example, then we want x bounded below by **one** function of y and z , and we want x bounded above by **one** function of y and z . If x is bounded above (or below) by two different functions of y and z , then we must express our integral as the sum of two integrals corresponding to the two different sets of bounds. We saw this situation arise in problem 32 of section 16.7 when trying to integrate first with respect to x .

- Once you have integrated with respect to the first variable, you are left with a double integral involving the other two variables. To evaluate the remaining double integral, we'd like to be able to sketch the 2D region we are integrating over, as we did before when evaluating double integrals. To accomplish this, we project the 3D region we are integrating over onto a plane. If we integrated

with respect to x first, we project B onto a 2D region in the yz plane; if we integrated with respect to y first, we project onto the xz plane, and so on.

QUESTION: How can you express the volume of a region, B , using a triple integral?

- **CYLINDRICAL AND SPHERICAL COORDINATES:** Sometimes it is easier to use polar coordinates to describe the 2D region of integration when evaluating a double integral. Likewise, sometimes it is easier to use cylindrical or spherical coordinates to describe the 3D region of integration when evaluating a triple integral. Again, you can decide which set of coordinates to use based on which coordinate system allows you to express the integrand and the 3D region you are integrating over most easily.

QUESTION: How do we define the variables (r, θ, z) and (ρ, θ, ϕ) ?

- As before, when CONVERTING from one coordinate system to another, we have three pieces of information to convert:
 - (1) To convert the bounds, sketch the 3D region you are integrating over, and express that region using r , θ , and z (for cylindrical coordinates), or ρ , θ , and ϕ (for spherical coordinates).
 - (2) Convert the integrand (the function itself which you are integrating over) by making the appropriate substitutions for x , y , and z .
 - (3) Convert the term dV . In cylindrical coordinates, $dV = r dz dr d\theta$, and in spherical coordinates, $dV = \rho^2 \sin \phi d\rho d\phi d\theta$.

Change of Variables

This topic may seem completely new, but you actually started doing changes of variables several sections ago. Converting from cartesian coordinates to polar, cylindrical, or spherical coordinates are all examples of changes of variables.

EXERCISE: Try computing the Jacobian $\partial(x, y)/\partial(r, \theta)$ for changing from cartesian to polar coordinates. Also compute the Jacobians $\partial(x, y, z)/\partial(r, \theta, z)$ (for changing from cartesian to cylindrical) and $\partial(x, y, z)/\partial(\rho, \theta, \phi)$ (for changing from cartesian to spherical). This should help motivate what we did earlier when we converted the dA term of a double integral into polar coordinates, and when we converted the dV term of a triple integral into cylindrical or spherical coordinates.

In general, to make a change of variables we follow the same steps as we did when converting from cartesian to polar, cylindrical, or spherical coordinates:

- (1) Convert the bounds of your integral by sketching the region over which you are integrating and expressing that region in terms of the new set of variables you want to use.
- (2) Convert your function by substituting for x and y (and z) in terms of your new variables.
- (3) Convert the term dA or dV . Previously we just took this conversion as given. With any arbitrary change of variables, you multiply by the Jacobian to obtain the correct conversion factor.