M346 First Midterm Exam, September 18, 2003

The exam is closed book, but you may have a single hand-written 8.5×11 crib sheet. There are 5 problems, each worth 20 points. The first four are calculational, and you MUST JUSTIFY YOUR ANSWERS. Part credit will be given, but answers without justification will not receive credit.

The fifth problem is a series of true/false questions. For these, you do not have to show your work, and part credit will NOT be given.

1. Let $M_{2,2}$ be the space of 2×2 real matrices, with standard basis $\mathcal{E} = \{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}$ and alternate basis $\mathcal{B} = \{\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 4 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}$. Let $A = \begin{pmatrix} 3 & 2 \\ 7 & 2 \end{pmatrix}$. Compute

a) the change-of-basis matrix $P_{\mathcal{E}\mathcal{B}}$,

$$P_{\mathcal{E}\mathcal{B}} = \begin{pmatrix} [\mathbf{b}_1]_{\mathcal{E}} & [\mathbf{b}_2]_{\mathcal{E}} & [\mathbf{b}_3]_{\mathcal{E}} & [\mathbf{b}_4]_{\mathcal{E}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & 4 & 1 \end{pmatrix}$$

b) the change-of-basis matrix $P_{\mathcal{B}\mathcal{E}}$,

$$P_{\mathcal{B}\mathcal{E}} = P_{\mathcal{E}\mathcal{B}}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -2 & 5 & -4 & 1 \end{pmatrix}$$

c) the coordinates of A in the $\mathcal{E}$ basis, and

By inspection, $[A]_{\mathcal{E}} = (3, 2, 7, 2)^T$.

d) the coordinates of A in the $\mathcal{B}$ basis.

$[A]_{\mathcal{B}} = P_{\mathcal{B}\mathcal{E}}[A]_{\mathcal{E}} = (3, -4, 6, -22)^T$.

2. Let $Z = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, and let $L : M_{2,2} \mapsto M_{2,2}$ be given by $L(M) = ZM$, where ZM is the matrix product of the 2×2 matrix Z and the 2×2 matrix M. L is a linear transformation. Find the matrix of L relative to the $\mathcal{E}$ basis of problem 1.

Since $L(e_1) = \begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix} = e_1 + 3e_3$, $L(e_2) = \begin{pmatrix} 0 & 1 \\ 0 & 3 \end{pmatrix} = e_2 + 3e_4$, $L(e_3) = \begin{pmatrix} 2 & 0 \\ 4 & 0 \end{pmatrix} = 2e_1 + 4e_3$, $L(e_4) = \begin{pmatrix} 0 & 2 \\ 0 & 4 \end{pmatrix} = 2e_2 + 4e_4$, we have

$$[L]_{\mathcal{E}} = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 4 & 0 \\ 0 & 3 & 0 & 4 \end{pmatrix}$$

3. Let $L : M_{2,2} \mapsto M_{2,2}$ be the linear transformation

$$L \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & a \\ d & b \end{pmatrix}$$

Compute

a) the matrix of L in the $\mathcal{E}$ basis of problem 1, and

Since $L(e_1) = e - 2$, $L(e_2) = e_4$, $L(e_3) = e_1$, $L(e_4) = e_3$, we have

$$[L]_{\mathcal{E}} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

b) the matrix of L in the $\mathcal{B}$ basis of problem 1.

$$[L]_{\mathcal{B}} = P_{\mathcal{B}\mathcal{E}}[L]_{\mathcal{E}}[L]_{\mathcal{B}} = \begin{pmatrix} 3 & 2 & 1 & 0 \\ -5 & -4 & -2 & 0 \\ 5 & 5 & 5 & 1 \\ -15 & -15 & -18 & -4 \end{pmatrix}$$

4. In $\mathbb{R}^2$, let $\mathbf{b}_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, $\mathbf{b}_2 = \begin{pmatrix} -5 \\ 3 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} -7 \\ 5 \end{pmatrix}$, $\mathbf{y} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Let W_1 be the line $x_1 + 2x_2 = 0$, and let W_2 be the line $3x_1 + 5x_2 = 0$.

a) Write $\mathbf{v}$ and $\mathbf{y}$ explicitly as linear combinations of $\mathbf{b}_1$ and $\mathbf{b}_2$.

$\mathbf{v} = 4\mathbf{b}_1 + 3\mathbf{b}_2$ and $\mathbf{y} = 3\mathbf{b}_1 + \mathbf{b}_2$, as can be seen in each case by solving 2 equations in 2 unknowns.

b) Viewing $\mathbb{R}^2$ as the (internal) direct sum of W_1 and W_2, compute $P_1\mathbf{v}$ (the projection of $\mathbf{v}$ onto W_1) and $P_2\mathbf{y}$.

Since W_1 is the span of $\mathbf{b}_1$ and W_2 is the span of $\mathbf{b}_2$, this is just asking for the $\mathbf{b}_1$ part of $\mathbf{v}$ and the $\mathbf{b}_2$ part of $\mathbf{y}$, namely $P_1\mathbf{v} = 4\mathbf{b}_1$ and $P_2\mathbf{y} = \mathbf{b}_2$.

c) Find the coordinates of $\mathbf{v}, P_1\mathbf{v}, \mathbf{y}$ and $P_2\mathbf{y}$ in the $\{\mathbf{b}_1, \mathbf{b}_2\}$ basis.

This is just restating the answers to parts (a) and (b), namely:
$[\mathbf{v}]_{\mathcal{B}} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$, $[P_1\mathbf{v}]_{\mathcal{B}} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$, $[\mathbf{y}]_{\mathcal{B}} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $[P_2\mathbf{y}]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

5. True of False? Each question is worth 4 points. You do NOT need to justify your answers, and partial credit will NOT be given.

a) The line $3x_1 + 5x_2 = 1$ is a subspace of $\mathbb{R}^2$.

False. This line does not contain the origin.

b) If A is a 3×5 matrix, then the dimension of the null space of A is at most 2.

False. The dimension is at LEAST 2, and can be more (e.g., if A is the the zero matrix, then the null space is 5 dimensional).

c) Every 5-dimensional subspace of $\mathbb{R}^8$ is the column space of a 8×5 matrix.

True. Take 5 basis vectors for this subspace and write them as the columns of a 8×5 matrix.

d) The vectors $\begin{pmatrix} 1 \\ 3 \\ 5 \\ 7 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, and $\begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$ span a 3-dimensional subspace of $\mathbb{R}^4$.

False. These vectors are linearly dependent and only span a 2-dimensional subspace of $\mathbb{R}^4$.

e) Let A be an $n \times m$ matrix. If there is only one solution to $A\mathbf{x} = 0$, then the columns of A are linearly independent.

True. See Theorem 2.3.