Algebraic Topology Final Exam

This is a self-timed, 3 hour, closed-book exam. Set aside 3 hours between now and Wednesday, December 9, do the test without any outside aid, and turn it in by the end of the day on Wednesday.

The test has six questions, of which you are expected to do four. Please do problem 1, problem 2, either problem 3 or problem 4 (your choice), and either problem 5 or problem 6. If you attempt both 3 and 4, or both 5 and 6, I will pick one of the problems arbitrarily to grade, and ignore the other one!

Good luck!!

1) Let X be the connected sum of two tori, let a_1 and b_1 be the meridian and longitude of the first torus, and let a_2 and b_2 be the meridian and longitude of the second torus. There is a simple closed curve γ that is homotopic to $a_1 b_1 a_1^{-1} b_1^{-1}$. Let Y be the union of X with a 2-disk D, where the boundary of D is identified with γ.

(a) Use van Kampen's theorem to compute $\pi_1(Y)$, where U and V are neighborhoods of X and D, respectively.

(b) Use Mayer-Vietoris, with the same open sets, to compute the homology of Y.

For both (a) and (b), you do not have to re-derive the fundamental group or homology of X. You can take the usual formulas as given.

2. Now let $X = \#_3 T^2$ be the connected sum of three tori. The universal cover of X is the hyperbolic plane, which is homeomorphic to $\mathbb{R}^2$.

(a) Show that any continuous map $f : RP^2 \to X$ is homotopic to a constant map.

(b) Describe a map $T^2 \to X$ that is not homotopic to a constant map, and prove that it is not homotopic.

3. Give a topological proof that a free group on n generators embeds in the free group on 2 generators, where n is an arbitrary positive integer.

4. Let Y_n be a chain with n links. More precisely, let $Y_n \subset \mathbb{R}^2$ be the union of the circles of diameter 1 centered at (1,0), (2,0), $\ldots$, (n,0). For what pairs of integers (m, n) is Y_m a covering space of Y_n? (For each pair of integers, either describe such a covering or show it does not exist.)

5. Let X be a topological space, and let $f : X \to X$ be a homeomorphism. The *mapping cylinder* of f, which I'll denote C_f, is the quotient of $[0, 1] \times X$ by the identifications $(1, x) \sim (0, f(x))$.

(a) Let $X = S^2$, and let f be the identity map. Compute the homology of C_f.

(b) Let $X = S^2$, and let f be the antipodal map. Compute the homology of C_f.

6. If $m < n$, the embedding of $\mathbb{R}^{m+1}$ in $\mathbb{R}^{n+1}$ induces an embedding of RP^m in RP^n. Compute the relative homology $H_k(RP^n, RP^m)$ for all k, n, m with $n > m$. Note that your answer may depend on the parity of m and n.