Algebraic Topology Final Exam Solutions

1) Let X be the connected sum of two tori, let a_1 and b_1 be the meridian and longitude of the first torus, and let a_2 and b_2 be the meridian and longitude of the second torus. There is a simple closed curve γ that is homotopic to $a_1 b_1 a_1^{-1} b_1^{-1}$. Let Y be the union of X with a 2-disk D, where the boundary of D is identified with γ.

(a) Use van Kampen's theorem to compute $\pi_1(Y)$, where U and V are neighborhoods of X and D, respectively.

(b) Use Mayer-Vietoris, with the same open sets, to compute the homology of Y.

For both (a) and (b), you do not have to re-derive the fundamental group or homology of X. You can take the usual formulas as given.

(a) Since $\pi_1(U) = \pi_1(X)$ and $\pi_1(V) = 0$, $\pi_1(Y)$ has the same generators as $\pi_1(X)$, only with one more relation, namely that the image of $\pi_1(U \cap V)$ (i.e., the cyclic group generated by $a_1 b_1 a_1^{-1} b_1^{-1}$) is zero. Thus $\pi_1(Y)$ is

$$< a_1, b_1, a_2, b_2 | [a_1, b_1], [a_1, b_1][a_2, b_2] >=< a_1, b_1, a_2, b_2 | [a_1, b_1], [a_2, b_2] >,$$

which equals $\mathbf{Z}^2 * \mathbf{Z}^2$. This answer can also be understood from the fact that $[a_1, b_1]$ is the curve that joins the two tori, so Y is homotopy equivalent to the wedge of two tori.

(b) From the Mayer-Vietoris sequence in reduced homology we have $0 \to H_2(S^1) \to H_2(X) \to H_2(Y) \to H_1(S^1) \to H_1(X) \to H_1(Y) \to 0$. The map $H_1(S^1) \to H_1(X)$ sends the generator of $H_1(S^1)$ to $a_1 + b_1 - a_1 - b_1 = 0$, so $H_1(Y) = H_1(X) = \mathbf{Z}^4$ and we have a split exact sequence $0 \to H_2(X) \to H_2(Y) \to \mathbf{Z} \to 0$, so $H_2(Y) = \mathbf{Z}^2$. $H_0(Y) = \mathbf{Z}$, of course, since everything is path-connected. Note that $H_1(Y)$ is the abelianization of $\pi_1(Y)$, as it must be.

2. Now let $X = \#_3 T^2$ be the connected sum of three tori. The universal cover of X is the hyperbolic plane, which is homeomorphic to $\mathbb{R}^2$.

(a) Show that any continuous map $f : RP^2 \to X$ is homotopic to a constant map.

Since $\pi_1(RP^2) = \mathbf{Z}_2$, the image of $\pi_1(RP^2)$ in $\pi_1(X)$ is either trivial or a torsion subgroup. However, $\pi_1(X)$ has no torsion elements (this takes some serious algebra to prove, but I'll give full credit for just stating this algebraic fact and extra credit for anybody who proved it). That means $\pi_1(X)$ goes to the trivial group, so f lifts to a map from RP^2 to the universal cover of X.

But this universal cover is contractible, so the lift is homotopic to a constant map, so f is homotopic to a constant map.

(b) Describe a map $T^2 \to X$ that is not homotopic to a constant map, and prove that it is not homotopic.

First map $T^2 = S^1 \times S^1$ to a circle in the obvious way, and then map that circle to a representative of any nontrivial element of $\pi_1(X)$. Calling the composite map $f : T^2 \to X$, we have that $f_* : \pi_1(T^2) \to \pi_1(X)$ has a nontrivial image, and so is not equal to the map on π_1 induced by a constant map $T^2 \to X$. But this means that f is not homotopic to a constant map, since homotopic maps give the same push-forwards.

As far as I know, there aren't any truly 2-dimensional maps $f : T^2 \to X$, in the sense that f_* sends $H_2(T^2)$ nontrivially to $H_2(X)$. I think that this obscure fact can be proven with cohomology. For your amusement, here's a sketch of the argument (with LOTS of details left out), which gives the flavor of cohomology as opposed to homology: There are lots of pairs (α, β) of pairs of classes in $H^1(X)$ whose product is the generator of $H^2(X)$. However, $f^* : \mathbf{Z}^4 = H^1(X) \to H^1(T^2) = \mathbf{Z}^2$ has a big kernel, so some of the elements in these pairs are in the kernel. But f^* is a ring homomorphism, so it must send the product $\alpha \cup \beta$ to zero, so $f^*(H^2(X))$ is trivial, so $f_*(H_2(T^2))$ is also trivial.

3. Give a topological proof that a free group on n generators embeds in the free group on 2 generators, where n is an arbitrary positive integer.

Let X_2 be the figure 8 curve. As we have seen several times, there is a cover X_n of X_2 that has the homotopy type of the wedge of n circles, for any integer $n > 1$. But that means that the fundamental group of X_n (namely the free group on n generators) injects in the fundamental group of X_2. As for $n = 1$, just take a map from the circle to the figure 8 that isn't homotopic to a constant map. This induces an injection of the free group on one generator into the fundamental group of X_2.

4. Let Y_n be a chain with n links. More precisely, let $Y_n \subset \mathbb{R}^2$ be the union of the circles of diameter 1 centered at (1,0), (2,0), ..., (n,0). For what pairs of integers (m, n) is Y_m a covering space of Y_n? (For each pair of integers, either describe such a covering or show it does not exist.)

It's easy to see that $H_1(Y_n) = \mathbf{Z}^n$, so the Euler characteristic of Y_n is $1 - n$. Since the Euler characteristic of Y_m must be a multiple of that of Y_n, we must have that $m - 1 = k(n - 1)$ for a positive integer k. Such k-fold covers do exist. The $m-1$-fold covering of Y_2 by Y_m was done in class. Think of Y_n as Y_2 with $n-2$ extra circles in the middle. Those extra circles come along for the ride in the construction of the cover.

When $k=2$, the cover is regular, since all index-2 subgroups are normal, and of course the trivial $k=1$ cover is regular. When $k>2$, there are no regular covers, because Y_m has two distinguished points, namely the first and last of the $m-1$ vertices. These are the only points that have paths to and from themselves without touching any other vertices first. Since in a regular cover the preimages of each vertex all look the same (and can be swapped by deck transformations), the image of one of these special points must have at most two preimages, so the regular cover of Y_n by Y_m can only be 1-fold or 2-fold.

5. Let X be a topological space, and let $f : X \to X$ be a homeomorphism. The mapping cylinder *of f, which I'll denote C_f, is the quotient of $[0,1]\times X$ by the identifications $(1,x) \sim (0, f(x))$.*

(a) Let $X = S^2$, and let f be the identity map. Compute the homology of C_f.

(b) Let $X = S^2$, and let f be the antipodal map. Compute the homology of C_f.

This is almost identical (but shifted in dimension) to a homework problem you did, comparing Mayer-Vietoris for a torus (i.e., the mapping cylinder of the identity map from a circle to itself) and a Klein bottle (the mapping cylinder of the reflection of a circle). Take U to be the region $1/4 < t < 3/4$ and let V be the region $\{t < 1/3 \text{ or } t > 2/3\}$. U and V both deformation retract to S^2, $U \cap V$ deformation retracts to two disjoint spheres, and we have $0 \to H_3(C_f) \to \mathbf{Z}^2 \to \mathbf{Z}^2 \to H_2(C_f) \to 0 \to 0 \to H_1(C_f) \to \mathbf{Z} \to 0 \to \tilde{H}_0(C_f) \to 0$. This gives $H_0 = H_1 = \mathbf{Z}$ and H_3 and H_2 are the kernel and cokernel of the map $\mathbf{Z}^2 \to \mathbf{Z}^2$ induced by inclusion of $U \cap V$ into U and into V.

After picking bases for $H_2(U\cap V)$, $H_2(U)$ and $H_2(V)$, the map is just a 2-by-2 matrix. When f is the identity, it is $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, with 1-dimensional kernel and 1-dimensional cokernel, so $H_3 = H_2 = \mathbf{Z}$. When f is the antipodal map, our matrix is $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ with trivial kernel and cokernel $\mathbf{Z}_2$, so $H_3 = 0$ and $H_2 = \mathbf{Z}_2$.

6. If $m < n$, the embedding of $\mathbb{R}^{m+1}$ in $\mathbb{R}^{n+1}$ induces an embedding of RP^m in RP^n. Compute the relative homology $H_k(RP^n, RP^m)$ for all k, n, m with

$n > m$. Note that your answer may depend on the parity of m and n.

(RP^n, RP^m) is a good pair, so we just have to compute the reduced homology of RP^n/RP^m. This is a CW complex with one 0-cell (representing all of RP^m) and one k-cell for each $m < k \leq n$, and the boundary map ∂_k is exactly as in RP^n: it is zero if k is odd or equal to $m+1$ and is multiplication by 2 if k is even and greater than $m+1$. Our homology groups $H_k(RP^n, RP^m)$ are therefore $\mathbf{Z}$ if $k = n$ and n is odd or $k = m + 1$ and $m + 1$ is even, is Z_2 for all odd $m < k < n$, and is zero for all other k.