M362K Third Midterm Exam, given on April 9, 2001

Problem 1. Monday Blues

A word is chosen at random from the (undoubtedly true) sentence "I hate to take math tests on Mondays". That is, each word has an equal chance of being chosen. Let X be the number of letters in the word. Let Y be the number of vowels in the word (yes, the "y" in Mondays counts as a vowel).

a) Write down the pdf of X (that is, $f_X(x)$).

There are 8 words in all, one with one letter and one vowel (I) , two with two letters and one vowel (to, on), one with four letters and one vowel (math), two with four letters and two vowels (hate, take), one with 5 letters and one vowel (tests) and one with 7 letters and 3 vowels (Mondays). Since each word has an equal probability, the joint pdf is given in the following table:

$Y\backslash X$	1	2	3	4	5	6	7	Total
1	1/8	2/8	0	1/8	1/8	0	0	5/8
2	0	0	0	2/8	0	0	0	2/8
3	0	0	0	0	0	0	1/8	1/8
Total	1/8	2/8	0	3/8	1/8	0	1/8	1

b) Compute the expectation $E(X)$.

$E(X) = \sum k f_X(k) = 1(1/8) + 2(2/8) + 4(3/8) + 5(1/8) + 7(1/8) = 29/8.$

c) Write down the joint pdf $f_{X,Y}(x, y)$.

(see above)

d) Are the events $X = 4$ and $Y = 2$ independent? Are the events $X = 6$ and $Y = 2$ independent? Are X and Y independent random variables?

P(X=4 and Y=2)=2/8, while P(X=4)=3/8 and P(Y=2)=2/8. Since 2/8 is not equal to 3/8 times 2/8, the events X=4 and Y=2 are not independent.

P(X=6)=0, P(Y=2)=2/8, and P(X=6 and Y=2)=0, which does equal 0 times 2/8, so these events ARE independent.

Since X=4 and Y=2 are not independent events, X and Y are not independent random variables.

Problem 2. Joint distributions

Let X and Y be independent continuous random variables, each chosen uniformly in the interval $[0, 1]$. That is,

$$f_{X,Y}(x, y) = \begin{cases} 1 & \text{if } 0 < x < 1 \text{ and } 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Let Z be the larger of X and Y. That is,

$$Z = \begin{cases} X & \text{if } X \geq Y \\ Y & \text{if } Y \geq X \end{cases}$$

a) What is the probability that ($X \leq 1/2$ and $Y \leq 1/2$)?

Since X and Y are independent, P($X < 1/2$ and $Y < 1/2$)=P($X < 1/2$) P($Y < 1/2$) = P$(X < 1/2)^2$ = $(1/2)^2 = 1/4$.

b) Compute the cumulative distribution function $F_Z(z)$.

Since Z is the larger of X and Y,

$$F_Z(z) = P(Z \leq z) = P(X \leq z \text{ and } Y \leq z) = P(X \leq z)^2 = \begin{cases} 0 & \text{if } z \leq 0 \\ z^2 & \text{if } 0 < z < 1 \\ 1 & \text{if } z > 1 \end{cases}$$

c) Compute the probability density function $f_Z(z)$.

$$f_Z(z) = dF_Z(z)/dz = \begin{cases} 2z & \text{if } 0 \leq z \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

d) Compute the expectation $E(Z)$.

$E(Z) = \int_{-\infty}^{\infty} z f_Z(z) dz = \int_0^1 2z^2 dz = 2/3$.

Problem 3. Reading CDFs

A random variable X has cumulative distribution function

$$F_X(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x^2/4 & \text{if } 0 < x < 1 \\ (1+x)/4 & \text{if } 1 \leq x < 2 \\ 1 & \text{if } x \geq 2 \end{cases}$$

(You may find it helpful to sketch this function).

a) What is the probability that $X = 0$? What is the probability that $X = 1$? What is the probability that $X = 2$?

P(X=0) = (jump in $F_X(x)$ at $x = 0$) = 0.
P(X=1) = (jump in $F_X(x)$ at $x = 1$) = 1/4.
P(X=2) = (jump in $F_X(x)$ at $x = 2$) = 1/4.

b) What is the probability that $1/2 < X \leq 1$?

P$(1/2 < X \leq 1) = F_X(1) - F_X(1/2) = 1/2 - 1/16 = 7/16$.

c) What is the probability that $1/2 \leq X < 1$?

This is the same as (b), plus the probability that $X = 1/2$ (which is zero), minus the probability that $X = 1$ (which is 1/4), that is 7/16 + 0 - 1/4 = 3/16.

d) What is the probability that $X > 1.5$

This is $1 - F_X(1.5) = 1 - 5/8 = 3/8$.

Problem 4. Manipulating random variables

Let X be continuously distributed between 1 and e with pdf

$$f_X(x) = \begin{cases} 1/x & \text{if } 1 < x < e \\ 0 & \text{otherwise} \end{cases}$$

a) Compute $E(X)$.

$E(X) = \int_1^e x f_X(x) dx = \int_1^e 1 dx = e - 1$.

b) Compute $F_X(x)$.

This is zero if $x \leq 1$ and one if $x \geq e$. If $1 < x < e$, then $F_X(x) = \int_1^x f_X(t) dt = \ln(x)$.

c) Let $Y = \ln(X)$. Compute $F_Y(y)$, and from it compute $f_Y(y)$.

$F_Y(y) = P(Y \leq y) = P(X \leq e^y) = F_X(e^y) = \begin{cases} 0 & \text{if } y \leq 0 \\ y & \text{if } 0 < y \leq 1 \\ 1 & \text{if } y > 1 \end{cases}$. In other words, Y is uniformly distributed between 0 and 1.

d) Compute $E(Y)$. [There are two ways to do this. One uses the results of part (b). The other does not.]

$E(Y) = \int_0^1 y f_Y(y) dy = \int_0^1 y dy = 1/2$, or

$E(Y) = E(\ln(X)) = \int_1^e \ln(x) f_X(x) dx = \int_1^e \frac{\ln(x) dx}{x} = (\ln(x))^2/2|_1^e = 1/2$.