M362K Third Midterm Exam Solutions. April 16, 2004

Problem 1. Manipulating continuous variables

Let X be a continuous random variable with pdf

$$f_X(x) = \begin{cases} 6x^{-7} & \text{if } x \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

a) Compute the expectation $E(X)$

$E(X) = \int_1^\infty x f(x) dx = \int_1^\infty 6x^{-6} dx = 6/5.$

b) Compute the variance $Var(X)$

$E(X^2) = \int_1^\infty x^2 f(x) dx = 6/4 = 3/2$, so $Var(X) = E(X^2) - (E(X))^2 = (3/2) - 36/25 = 3/50 = .06$

c) Let $Y = X^2$. Compute the pdf $f_Y(y)$. Since $X = \sqrt{Y}$,

$f_Y(y) = f_X(\sqrt{y})/|dy/dx| = f_X(\sqrt{y})/2\sqrt{y} = 3y^{-4}$.

Problem 2. Continuous joint distributions

Let X and Y be continuous random variables with joint pdf

$$f_{X,Y}(x, y) = \begin{cases} xe^{-(x+y)} & \text{if } x > 0 \text{ and } y > 0, \\ 0 & \text{otherwise} \end{cases}$$

a) Are X and Y independent random variables? Why or why not?

Yes, since $f_{X,Y}(x, y)$ factors into a function of x (xe^{-x}) times a function of y (e^{-y}) and since the domain is rectangular. It's not hard to compute $f_X(x) = xe^{-x}$ when $x >$ and $f_Y(y) = e^{-y}$ when $y > 0$.

b) Let $Z = X + Y$. Find the cdf $f_Z(z)$ for all values of z.

$f_Z(z) = \int_{-\infty}^\infty f_{X,Y}(x, z - x) dx$. This integral is zero if $z \leq 0$, but equals $\int_0^z xe^{-z} dx = z^2 e^{-z}/2$ for $z > 0$.

Problem 3. A dicey problem Two fair dice are rolled. Let X be the value of the higher die, and let Y be the value of the lower die. (If the two dice give the same value, say double 4's, then both X and Y would equal 4, while if we got a 5 and a 3 we would have $X = 5$ and $Y = 3$).

a) Find the joint pdf $f_{X,Y}(x, y)$ for all posible pairs (x, y).

For each pair (x, y) with $x < y$, the probability is zero, if $x > y$ the probability is 2/36, (since we could get roll x and y or y and x), and if $x = y$ the probability is 1/36.

b) Compute the marginal pdf's $f_X(x)$ and $f_Y(y)$.

The entire situation is summarized in the table:

	$X = 1$	2	3	4	5	6	f_Y
$Y = 1$	1/36	2/36	2/36	2/36	2/36	2/36	11/36
2	0	1/36	2/36	2/36	2/36	2/36	9/36
3	0	0	1/36	2/36	2/36	2/36	7/36
4	0	0	0	1/36	2/36	2/36	5/36
5	0	0	0	0	1/36	2/36	3/36
6	0	0	0	0	0	1/36	1/36
f_X	1/36	3/36	5/36	7/36	9/36	11/36	1

c) Find $F_X(3)$.

$F_X(3) = f_X(1) + f_X(2) + f_X(3) = 1/36 + 3/36 + 5/36 = 9/36 = 1/4.$

Problem 4. Lottery tickets A lottery is designed so that each ticket has a 10% change of paying \$ 2, a 4% chance of paying \$ 5, a 1% chance of paying \$ 10, and an 85% chance of paying nothing. You buy a ticket, and call its value X.

a) What is the expectation $E(X)$?

$E(X) = 0(0.85) + 2(0.1) + 5(0.4) + 10(0.01) = 0.5$

b) Compute the variance $Var(X)$ and the standard deviation σ_x.

$E(X^2) = 0^2(0.85) + 2^2(0.1) + 5^2(0.4) + 10^2(0.01) = 2.4$, so $Var(X) = 2.4 - (0.5)^2 = 2.15$. $\sigma_x = \sqrt{Var(X)} = 1.466$.

c) Suppose you buy 100 lottery tickets, where each ticket is independent of the others. Let Y be the total value of all 100 tickets put together. Compute $E(Y)$ and σ_y. Since expectations scale as N^1 and standard deviations scale as $\sqrt{N}$,

$E(Y) = 100E(X) = 50$ and $\sigma_y = \sqrt{100}\sigma_x = 14.66$.