M362K Probability Homework Solutions
Homework 1: Due September 1

Chapter 1 problems, Page 16-17, # 3, 7, 12, 13, 21, 22, 24.

Chapter 1 theoretical exercises, page 18-19, # 3, 12a, 13, 16

Problems

Problem 3. There are 20 ways to assign the first worker, 19 ways to assign the second (since you can't give him the first guy's job), 18 ways to assign the third, and so on, so there are 20! possibilities in all.

Problem 7. a) 6!=720, as you are just arranging 6 people in order.

b) There are $3! = 6$ ways to arrange the boys, $3! = 6$ ways to arrange the girls, and 2 choices of whether the girls are to the left or right of the boys, so the total is $6 \times 6 \times 2 = 72$.

c) Now there are 6 ways to arrange the boys among themselves and $4! = 24$ ways to arrange the four groups (with each girl being her own group), for $6 \times 24 = 144$ ways in all.

d) This is like part b, since you can either give the boys seats 1,3,5 or seats 2,4,6, and either choice leaves 3! ways to arrange the boys and 3! ways to arrange the girls.

Problem 12. a) 30^5, since each of the 5 awards can be given in 30 different ways. b) $30 \times 29 \times 28 \times 27 \times 26 = 30!/25!$, since there are 30 ways to give out the first award, then 29 ways to give out the second, and so on.

Problem 13. 20 people each shake hands with 19 other people, so you might think the answer would be 380. However, this is counting every handshake twice, once from the perspective of each person, so the answer is *half* that, or $190 = \binom{20}{2}$.

Problem 21. You take 7 steps, of which 3 are vertical. You have to choose which 3, so there are $\binom{7}{3} = 35$ ways to do it. (You could also choose the 4 horizontal steps to get $\binom{7}{4} = 35$.)

Problem 22. There are $\binom{4}{2} = 6$ ways to get from A to the circled point, then $\binom{3}{1} = 3$ ways to get from there to B, so the total is $6 \times 3 = 18$.

Problem 24. $(3x^2+y)^5 = (3x^2)^5+5(3x^2)^4(y)^1+10(3x^2)^3y^2+10(3x^2)^2y^3+5(3x^2)y^4+y^5 = 243x^{10} + 405x^8y + 270x^6y^2 + 90x^4y^3 + 15x^2y^4 + y^5$.

Theoretical Exercises

Exercise 3. There are n ways to pick the first one, $n-1$ ways to pick the second, etc, so the answer is $n!/(n-r)!$.

Exercise 12a. There are $\binom{n}{k}$ ways to choose a committee of k, and then k ways to choose its chair, so there are $k\binom{n}{k}$ ways to choose a k-person committee and a chairperson, and $\sum_k k\binom{n}{k}$ ways to choose a committee of arbitrary size and a chairperson. On the other hand, there are n ways to first choose the committee chair, and then 2^{n-1} ways to decide whether each of the remaining $n-1$ people should or shouldn't be on the committee. Comparing these two calculations gives $n2^{n-1} = \sum_{k=1}^{n} k\binom{n}{k}$.

The identity $n2^{n-1} = \sum_{k=1}^{n} k\binom{n}{k}$ can be derived a completely different way, using the binomial theorem. Since $(1+x)^n = \sum_k x^k\binom{n}{k}$ for *every* value of x, it's an equality of two functions, so we can take the derivatives of both sides to get $n(1+x)^n = \sum_k k\binom{n}{k}x^{k-1}$. Now plug in $x=1$ to get $n2^{n-1} = \sum_k k\binom{n}{k}$. As with much of combinatorics, there's more than one way to skin a cat.

Exercise 13. If $n > 0$, then $0 = (-1+1)^n = \sum_{i=0}^{n}\binom{n}{i}(-1)^k 1^{n-i} = \sum_{i=0}^{n}(-1)^i\binom{n}{i}$.

Exercise 16. Let $s_1, \ldots, s_n$ be the scores of the n contestants. a) There are 13 possibilities, including six where there are no ties (that is, $s_1 > s_2 > s_3$ and all permutations of the three contestants), 3 where there's a 2-way tie for first (e.g., $s_1 = s_2 > s_3$, where there are 3 possibilities for who comes in last), 3 where there's a 2-way tie for last (e.g., $s_1 > s_2 = s_3$), and one with a 3-way tie ($s_1 = s_2 = s_3$).

b) We arrange the situation for n contestants by first considering the number i of people who come in last (anywhere from 1 to n people), then picking these i people ($\binom{n}{i}$ possibilities), and then arranging the $n-i$ others ($N(n-i)$ possibilities, noting that if $i=n$ there is nothing left to choose, hence $N(0)=1$ way to do it!), to get $\sum_{i=1}^{n}\binom{n}{i}N(n-i)$.

c) First let $k = n-i$, so $N(n) = \sum_{k=0}^{n-1}\binom{n}{n-k}N(k)$. But $\binom{n}{n-k} = \binom{n}{k}$, so $N(n) = \sum_{k=0}^{n-1}\binom{n}{k}N(k) = \sum_{i=0}^{n-1}\binom{n}{i}N(i)$, where in the last step we just changed the name of our dummy variable from k to i.

d) $N(3) = \binom{3}{0}N(0) + \binom{3}{1}N(1) + \binom{3}{2}N(2) = 1\times 1 + 3\times 1 + 3\times 3 = 13$, as we saw in part a. Likewise, $N(4) = N(0) + 4N(1) + 6N(2) + 4N(3) = 1 + 4 + 18 + 52 = 75$.