M362K Second Midterm Exam Solutions, November 3, 2010

1. Baseball

Major league baseball is considered switching to a shorter regular season with an expanded playoff system. Suppose that in the new league, the season runs for 144 games, and that team A has a 50% chance of winning each game it plays, independent of all the other games.

a) (10 pts) Approximate the probability that team A wins 80 games or more.

Let X be the number of games won by team A. This variable is binomial, with $n = 144$ and $p = 1/2$, hence mean $\mu = np = 72$, variance $\sigma^2 = np(1-p) = 36$ and standard deviation 6. Using the normal approximation, and including the continuity correction, we have $P(X \geq 80) = P(X > 79.5) = P(Z \geq \frac{79.5-72}{6}) = P(Z \geq 1.25) \approx 1 - \Phi(1.25) = 1 - 0.8944 = 0.1056$.

b) (5 pts) Give an *exact* expression for the probability that team A wins exactly 72 games.

Since this is binomial, the exact expression is $\binom{144}{72}/2^{144}$. This is, of course, impossible for a normal person to turn into a percentage.

c) (10 pts) Give a good numerical approximation for the probability that team A wins exactly 72 games. (Your answers to (a) and (c) should be numbers like 0.135, not expressions like $\binom{8}{3}e^{-5.4}$.)

$P(X = 72) = P(71.5 < X < 72.5) = P(-\frac{1}{12} < Z < \frac{1}{12}) \approx 2\Phi(0.08333) - 1 \approx 2(0.5332) - 1 = 0.0664$. The number 0.5332 is one third of the way from $\Phi(0.08)$ to $\Phi(0.09)$ in the table.

2. Being nickled and dimed.

Suppose that I independently flip three nickels and three dimes. The nickels are weighted so that they each have a 2/3 chance of coming up heads. The dimes are weighted so that they each have a 1/3 chance of coming up heads. Let X be the number of nickels that come up heads, let Y be the number of dimes that come up heads, and let $Z = X + Y$.

a) (9 pts) Compute the probability mass function p_X, the expectation $E(X)$ and the variance $Var(X)$.

X is a binomial random variable with $n = 3$ and $p = 2/3$, so $p_X(k) = \binom{3}{k}\left(\frac{2}{3}\right)^k\left(\frac{1}{3}\right)^{3-k}$. Explicitly, $p_X(0) = 1/27$, $p_X(1) = 6/27$, $p_X(2) = 12/27$, and $p_X(3) = 8/27$. The expectation and variance are $E(X) = np = 2$ and $Var(X) = np(1-p) = 2/3$.

b) (6 pts) Compute the joint probability mass function $p_{X,Y}$.

The distribution of Y is similar, only with $p = 1/3$ instead of $2/3$. Specifically, $p_Y(0) = 8/27$, $p_Y(1) = 12/27$, $p_Y(2) = 6/27$, and $p_Y(3) = 1/27$. Since X and Y are independent, $P_{X,Y}(a, b) = p_X(a)p_Y(b)$. This is expressed in the following table:

$Y\backslash X$	0	1	2	3	Total
0	$\frac{8}{729}$	$\frac{48}{729}$	$\frac{96}{729}$	$\frac{64}{729}$	$\frac{8}{27}$
1	$\frac{12}{729}$	$\frac{72}{729}$	$\frac{144}{729}$	$\frac{96}{729}$	$\frac{12}{27}$
2	$\frac{6}{729}$	$\frac{36}{729}$	$\frac{72}{729}$	$\frac{48}{729}$	$\frac{6}{27}$
3	$\frac{1}{729}$	$\frac{6}{729}$	$\frac{12}{729}$	$\frac{8}{729}$	$\frac{1}{27}$
Total	$\frac{1}{27}$	$\frac{6}{27}$	$\frac{12}{27}$	$\frac{8}{27}$	1

c) (10 pts) Compute the probability mass function p_Z.

$p_Z(0) = p_{X,Y}(0, 0) = 8/729$, $p_Z(1) = p_{X,Y}(0, 1) + p_{X,Y}(1, 0) = 60/729$. $p_Z(2) = p_{X,Y}(0, 2) + p_{X,Y}(1, 1) + p_{X,Y}(2, 0) = 174/729$. $p_Z(3) = p_{X,Y}(0, 3) + p_{X,Y}(1, 2) + p_{X,Y}(2, 1) + p_{X,Y}(3, 0) = 245/729$. $p_Z(4) = p_{X,Y}(1, 3) + p_{X,Y}(2, 2) + p_{X,Y}(3, 1) = 174/729$. $p_Z(5) = p_{X,Y}(2, 3) + p_{X,Y}(3, 2) = 60/729$. $p_Z(6) = p_{X,Y}(3, 3) = 8/729$.

3. A continuous distribution

Suppose that the pdf of a variable X is given by

$$f_X(x) = \begin{cases} 0 & \text{if } x < -1 \\ \frac{1}{2} & \text{if } -1 < x < 0 \\ \frac{e^{-x}}{2} & \text{if } x \geq 0 \end{cases}$$

Compute the cdf $F_X(x)$, the expectation $E(X)$ and the variance $Var(X)$. You may find the identity $\int_0^\infty x^n e^{-x} dx = n!$ to be useful.

$$F_X(a) = \int_{-\infty}^{a} f_X(x)dx = \begin{cases} 0 & a \leq -1 \\ \frac{a+1}{2} & -1 \leq a \leq 0 \\ \frac{1}{2}(2 - e^{-a}) & a \geq 0 \end{cases}$$

Don't forget the constants of integration!

$E(X) = \int_{-\infty}^{\infty} x f_X(x)dx = \int_{-1}^{0} \frac{x}{2} dx + \int_0^\infty \frac{xe^{-x}}{2} dx = -\frac{1}{4} + \frac{1}{2} = \frac{1}{4}$.

$E(X^2) = \int_{-1}^{0} \frac{x^2}{2} dx + \int_0^\infty \frac{x^2 e^{-x}}{2} dx = \frac{1}{6} + 1 = \frac{7}{6}$. Note that $\int_{-1}^{0} \frac{x}{2} dx = \frac{x^2}{4}|_{-1}^{0}$ is negative, since $x/2$ is negative, and that $\int_{-1}^{0} \frac{x^2}{2} dx = \frac{x^3}{6}|_{-1}^{0}$ is positive, since $x^2/2$ is positive. A remarkable number of people got that backwards.

$Var(X) = E(X^2) - (E(X))^2 = \frac{7}{6} - \frac{1}{16} = \frac{53}{48}$.

4. Fishing for answers

In a lake by the French Mathematical Institute, fishing is a Poisson process, with each fisherman catching an average of 0.5 fish per hour.

(a) (15 pts) If Pierre spends 10 hours fishing one day, what is the probability of his coming home without any fish? What is the probability that he will come home with 3 or more fish?

This is Poisson (which is also the French word for fish) with $\lambda = (0.5/\text{hr})(10$ hrs$)=5$, so the probability of no fish is $e^{-5} \approx 0.006738$. Call this number α. Meanwhile, $P(X \geq 3) = 1 - p_X(0) - p_X(1) - p_X(2) = 1 - e^{-5} - 5e^{-5} - \frac{25}{2}e^{-5} = 1 - \frac{37}{2}e^{-5}$.

(b) (10 pts) If 100 fishermen each spend 10 hours fishing, what is the (approximate) probability that at least two of them will come home without fish? (You may wish to express the answer to part (b) in terms of your answers to part (a)).

Strictly speaking the number of hungry fishermen is binomial with $p = \alpha$ and $n = 100$, so the probability of having k hungry fishermen is $\binom{100}{k}\alpha^k(1-\alpha)^{100-k}$, and the probability of having two or more hungry is $1-(1-\alpha)^{100} - 100\alpha(1-\alpha)^{99}$. That's a correct answer, and I gave full credit for it, but it's a real pain to calculate.

This is well approximated by Poisson with $\lambda = 100\alpha \approx 0.6738$. The probability of two or more hungry fishermen is then (approximately) $1 - e^{-100\alpha} - 100\alpha e^{-100\alpha} \approx 1 - (1.6738)e^{-.6738} \approx 0.1467$. Even though each individual fisherman would have to be very unlucky to come home empty handed, with 100 fishermen out there, there's a decent (15%) chance that two or more will not have any dinner.

The motivation for this problem was thinking about the Chicago Cubs, who haven't won a World Series in 102 years. The chance of any one particular team going a century without winning the Series is extremely small, but the chances of having at least one team do it is not that small.