M362K Third Midterm Exam Solutions, December 3, 2010

1. Heads I win ...

Suppose that we flip a fair coin 20 times. Let X be the number of heads in the first 15 flips. Let Y be the number of heads in the last 10 flips. Let $Z = 2X + 3Y$.

a) (15 pts) Compute $E(X)$, $E(Y)$, $Var(X)$, $Var(Y)$, and the covariance $Cov(X,Y)$.

Since X and Y are both binomial random variables with $p = 1/2$, we have $E(X) = np = 15/2$, $Var(X) = np(1-p) = 15/4$, $E(Y) = 5$, and $Var(Y) = 5/2$. If we let X_i be the result of the ith flip, then $X = X_1 + \cdots + X_{15}$ and $Y = X_{11} + \cdots + X_{20}$. Since the covariance of X_i and X_j is 1/4 if $i = j$ and 0 if $i \neq j$, $Cov(X,Y) = \sum_{i=1}^{15} \sum_{j=11}^{20} Cov(X_i, X_j) = 5/4$.

b) (10 pts) Compute the expectation and variance of Z.

$E(Z) = 2E(X) + 3E(Y) = 30$. $Var(Z) = 4Var(X) + 9Var(Y) + 12Cov(X,Y) = 15 + (45/2) + 15 = 105/2$.

2. Las Vegas rules.

At a certain casino, there is a game where you have a 50% chance of winning \$1, a 25% chance of breaking even, and a 25% chance of losing \$4.

a) (10 pts) Let X be your net winning (i.e. winnings minus losses) on one play of the game. Compute $E(X)$ and and the variance $Var(X)$.

$E(X) = .5(1) + .25(0) + .25(-4) = -0.5$. $E(X^2) = .5(1) + .25(0) + .25(16) = 9/2$, and $Var(X) = E(X^2) - (E(X))^2 = 17/4$.

b) (15 pts) Suppose that you play the game 68 times. Estimate the probability that you will come out ahead.

Let Y be the results of playing 68 times. Then $Var(Y) = 68Var(X) = 17^2$ and the standard deviation is 17, while $E(Y) = 68E(X) = -34 = -2\sigma$. By the central limit theorem, the distribution of Y is approximately normal. The probability of coming out ahead is the probability of coming out two or more standard deviations better than average, which is $1 - \Phi(2) = .0228$. Like most Las Vegas games, you're overwhelmingly likely to go home a loser.

3. Baseball.

Recall that, for a geometric random variable with parameter p, the expectation, variance and moment generating function are $1/p$, $(1-p)/p^2$, and $pe^t/[1-(1-p)e^t]$, respectively.

a) (12 points) Let X be a geometric random variable with $p = 0.1$, and let $Y = 3X + 5$. Compute $E(Y)$, $Var(Y)$ and $M_Y(t)$.

$E(X) = 10$ and $Var(X) = 0.9/(0.1)^2 = 90$, so $E(Y) = 3E(X) + 5 =$

$3(10) + 5 = 35$. $Var(Y) = 3^2 Var(X) = 810$. $M_Y(t) = e^{5t} M_X(3t) = .1e^{8t}/[1 - 0.9e^{3t}]$.

b) (13 points) A slugger has a 10% chance of hitting a home run every time he comes up to bat. (You can treat each plate appearance as an independent event.) How many at-bats does he need to be 99% sure of hitting 40 home runs (or more).

The number of at-bats needed to hit 40 home runs is the sum of 40 identically distributed random variables, each representing the number of at-bats needed to hit the next home run. Thus, if Z is the number required to hit 40 home runs, $E(Z) = 40E(X) = 400$ and $Var(X) = 40Var(X) = 3600$, so the standard deviation of Z is 60. By the central limit theorem, Z is approximately normal, and so has a 99.01% chance of being under $\mu + 2.33\sigma = 400 + 2.33(60) = 400 + 140 = 540$. In other words, 540 at-bats is enough to guarantee 40 home runs with 99% certainty.

4. And heads I win some more...

Imagine a game where you flip 10 fair coins. You get \$1 for the second heads, \$2 for the third heads, \$3 for the fourth heads, and so on. You get nothing for the first heads or for any coin that comes up tails. So, for instance, if you get 4 heads and 6 tails, you get paid \$ 0+1+2+3 = \$6. Let X be your total winnings. The object of this problem is to compute the expectation of X, step by step, and you **do** get credit for each step.

(a) (10 pts) Define random variables $X_1, \ldots, X_{10}$ such that $X = X_1 + \cdots + X_{10}$ and such that the probability distribution of each X_i is reasonable to understand.

Let X_i be the winnings from the i-th toss.

(b) (10 pts) Compute the expectations $E(X_i)$ of each X_i.

If the i-th toss is tails, $X_i = 0$. If the i-th toss is heads, then X_i is the number of previous heads, which averages out to $(i-1)/2$. Thus $E(X_i) = (i-1)/4$.

(c) (5 pts) Compute $E(X)$.

$E(X) = (0 + 1 + 2 + \cdots + 9)/4 = 45/4$, or \$11.25. Note that this is NOT the same as the payoff from getting an average number of heads, which is just \$10.

Extra credit (5 pts). Imagine that you flip n coins, rather than 10. Compute $E(X)$ as a function of n.

$E(X) = \sum_{i=1}^{n} (i-1)/4 = n(n-1)/8$.