Algebraic Topology Final Exam

This is a self-timed, 3 hour, closed-book exam. Set aside 3 hours between now and Wednesday, December 8, do the test without any outside aid, and turn it in by the end of the day on Wednesday.

The test has six questions, of which you are expected to do four. Please do problem 1, problem 2 and any other two problems. Please indicate clearly which problems you are attempting! If I see solutions to more than 2 other problems, I will grade the first two and ignore the rest.

Good luck!!

1) Let X be a CW complex consisting of one vertex p, 2 edges a and b, and two 2-cells f_1 and f_2, where the boundaries of a and b map to p, where the boundary of f_1 is the loop ab^2 (that is, first a and then b twice), and where the boundary of f_2 is the loop ba^2. Compute the fundamental group of X and the homology groups of X.

2) Let G be the free group on two generators a and b. Show that there exists a finitely-generated subgroup H of G of index 3 that is not normal. Give explicit generators for H.

3) Let X be a chain with an even number of links (say, viewed as circles of radius 1 in the x-y plane, with centers on the x axis spaced 2 apart) and let $r : X \to X$ be rotation by 180 degrees about the midpoint of X. Show that any map $f : X \to X$ that is homotopic to r has a fixed point.

4) Let $X_{g,n}$ be the orientable genus-g surface with n points removed, where $n > 0$. Compute the fundamental group and the first homology of $X_{g,n}$.

5) Let X be the 2-sphere with the north and south poles identified. Give a CW decomposition of X and use this to compute the homology of X.

6) Let $\{G_i\}$ be a family of groups, where the index set I is arbitrary. For each pair $i, j \in I$, let F_{ij} be a (possibly empty) set of homomorphisms $G_i \to G_j$. We then define a category as follows:

An object is a group G together with maps $\phi_i : G_i \to G$ such that, if $f_{ij} \in F_{ij}$, then $\phi_j \circ f_{ij} = \phi_i$. If $(G, \{\phi_i\})$ and $(G', \{\phi'_i\})$ are two such objects, then a morphism is a map $\psi : G \to G'$ such that, for each i, $\phi'_i = \psi \circ \phi_i$.

Identify the universal object of this category in the following four circumstances. In each case, you should explain your reasoning, but you do *not* have to give a complete proof that your answer has the universal property:

(A) When all the families F_{ij} are empty.

(B) When there is a single group $G_1 = Z$ and a single map $f \in F_{11}$ that is multiplication by an integer n.

(C) When there are three groups $G_{1,2,3}$ and the only nonempty families are F_{31} and F_{32}, each of which consists of a single injection of G_3 into G_1 or G_2.

(D) When the index set is the positive integers each $F_{i,i+1}$ consists of a single map $f_i : G_i \to G_{i+1}$, and all other F_{ij}'s are empty.