Algebraic Topology Final Exam Solutions

1) Let X be a CW complex consisting of one vertex p, 2 edges a and b, and two 2-cells f_1 and f_2, where the boundaries of a and b map to p, where the boundary of f_1 is the loop ab^2 (that is, first a and then b twice), and where the boundary of f_2 is the loop ba^2. Compute the fundamental group of X and the homology groups of X.

Back in homework 6 we examined the effect that gluing in a disk has on the fundamental group of a space. If X is obtained from Y by gluing in a disk, then $\pi_1(X)$ is the quotient of $\pi_1(Y)$ by the subgroup generated by the boundary of the disk.

In this case, we first consider the 1-skeleton of X, which has fundamental group F_2, with generators a and b. We then glue in f_1, which means modding out by the cyclic group generated by ab^2. We then glue in f_2 and mod out by ba^2. That is, $\pi_1(X) = \langle a, b | ab^2, ba^2 \rangle$. If we use the second relation to set $b = a^{-2}$, we get $\pi_1(X) = \langle a | a^{-3} \rangle = \mathbf{Z}_3$.

Now for homology. Using cellular homology, $C_2 = C_1 = \mathbf{Z}^2$, and the boundary map is $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$. This is a nonsingular matrix with determinant 3, so the kernel is trivial and the cokernel is $\mathbf{Z}_3$. Thus $H_2 = 0$ and $H_1 = \mathbf{Z}_3$ (and of course $H_0 = \mathbf{Z}$.)

2) Let G be the free group on two generators a and b. Show that there exists a finitely-generated subgroup H of G of index 3 that is not normal. Give explicit generators for H.

G is the fundamental group of a figure-8, so this is equivalent to finding an irregular triple cover of the figure 8. That's a chain $\tilde{X}$ with 4 links, which we can picture as a graph with three vertices p_1, p_2 and p_3, and edges a_1 from p_1 to itself, b_1 from p_1 to p_2, b_2 from p_2 to p_1, a_2 from p_2 to p_3, a_3 from p_3 to p_2, and b_3 from p_3 to itself.

The fundamental group $\pi_1(\tilde{X}, p_2)$ is a free group on the generators $b_2a_1b_2^{-1}$, b_2b_1, a_2a_3 and $a_2b_3a_2^{-1}$. This maps to the subgroup $H = \langle bab^{-1}, b^2, a^2, aba^{-1} \rangle$.

Looking at the fundamental group of $\tilde{X}$ based at p_1 or p_3 instead of p_2 would give different index-3 subgroups. One alternate answer is generated by a, b^2, ba^2b^{-1}, and $baba^{-1}b^{-1}$ and equals bHb^{-1}, while the other is generated by b, a^2, ab^2a^{-1} and $abab^{-1}a^{-1}$ and equals aHa^{-1}.

3) Let X be a chain with an even number of links (say, viewed as circles of

radius 1 in the x-y plane, with centers on the x axis spaced 2 apart) and let $r : X \to X$ be rotation by 180 degrees about the midpoint of X. Show that any map $f : X \to X$ that is homotopic to r has a fixed point.

Viewing X as a CW complex with $2n$ links, hence $2n-1$ vertices and $4n-2$ edges (much like the solution to problem 2), the map r takes exactly one vertex to itself and doesn't take any edges to themselves. Thus the trace of $r_\#$ is 1 on C_0 and zero on C_1, so the Lefschetz number of r (and the Lefschetz number of any map homotopic to r) is 1. Since this isn't zero, any map homotopic to r has a fixed point.

4) Let $X_{g,n}$ be the orientable genus-g surface with n points removed, where $n > 0$. Compute the fundamental group and the first homology of $X_{g,n}$.

We can picture $X_{g,n}$ as a $4g$-gon with edges identified, with n points removed from the interior of the $4g$-gon. Let U be the interior of the polygon, with fundamental group F_n and generators $c_1, \ldots, c_n$, where each c_i is a loop around a hole, connected by a path to our base point. Let V be a neighborhood of the boundary of the polygon, with fundamental group F_{2g} and generators $a_1, \ldots, a_g, b_1, \ldots, b_g$. The intersection is an annulus, so that $\pi_1(U \cap V) = \mathbf{Z}$, with a generator that goes once around the boundary of the polygon. Viewed in U, this gives $\prod c_i$. Viewed in V it gives $\prod [a_i, b_i]$. Thus we have

$$\pi_1(X_{g,n}) = \langle \{a_i\}, \{b_i\}, \{c_j\} | \prod_i [a_i, b_i] (\prod_j c_j)^{-1} \rangle.$$

We can use the relation to eliminate a single c_j, leaving us with the free group on $2g + n - 1$ generators (g a's, g b's and $n-1$ surviving c's.)

To get H_1, we can either abelianize π_1 (obtaining $\mathbf{Z}^{2g+n-1}$), or we can apply Mayer-Vietoris (most easily with reduced homology) to U and V. Since $\tilde{H}_1(U) = \mathbf{Z}^n$ and $\tilde{H}_1(V) = \mathbf{Z}^{2g}$ and $\tilde{H}_1(U \cap V) = \mathbf{Z}$ and everything is connected, we have

$$0 \to H_2(X) \to \mathbf{Z} \xrightarrow{i} \mathbf{Z}^{2g+n} \to H_1(X) \to 0.$$

All that is left is to identify $i(1)$, which is the image of the loop around the polygon in both U and V. In U it is $\sum c_j$, and in V is is $\sum_i (a_i + b_i - b_i - a_i) = 0$. Thus $i(1) = \sum c_j$, the kernel of i is trivial and the cokernel of i is $\mathbf{Z}^{2g+n-1}$. Thus $H_2(X) = 0$ and $H_1(X) = \mathbf{Z}^{2g+n-1}$.

5) Let X be the 2-sphere with the north and south poles identified. Give a CW decomposition of X and use this to compute the homology of X.

We have a CW composition with one vertex, one edge and one face. Let p be the north pole, which is also the south pole. Let e be the prime meridian, running from the north pole to the south pole. Let f be the image of a square, where the x coordinate gives longitude (say, starting at the prime meridian and running west) and the y coordinate gives latitude (say, with increasing y meaning going farther south).

The boundary of e is trivial, since the beginning and end points are identified. The boundary of f is a constant map at the north pole, a path along e, a constant map at the south pole, and a path along e backwards. Since we traverse e twice, once in each direction, the map ∂_2 is zero, so $H_2 = C_2 = \mathbf{Z}$, $H_1 = C_1 = \mathbf{Z}$ and $H_0 = C_0 = \mathbf{Z}$.

6) Let $\{G_i\}$ be a family of groups, where the index set I is arbitrary. For each pair $i, j \in I$, let F_{ij} be a (possibly empty) set of homomorphisms $G_i \to G_j$. We then define a category as follows:

An object is a group G together with maps $\phi_i : G_i \to G$ such that, if $f_{ij} \in F_{ij}$, then $\phi_j \circ f_{ij} = \phi_i$. If $(G, \{\phi_i\})$ and $(G', \{\phi_i'\})$ are two such objects, then a morphism is a map $\psi : G \to G'$ such that, for each i, $\phi_i' = \psi \circ \phi_i$.

Identify the universal object of this category in the following four circumstances. In each case, you should explain your reasoning, but you do *not* have to give a complete proof that your answer has the universal property:

(A) When all the families F_{ij} are empty.
(B) When there is a single group $G_1 = Z$ and a single map $f \in F_{11}$ that is multiplication by an integer n.
(C) When there are three groups $G_{1,2,3}$ and the only nonempty families are F_{31} and F_{32}, each of which consists of a single injection of G_3 into G_1 or G_2.
(D) When the index set is the positive integers each $F_{i,i+1}$ consists of a single map $f_i : G_i \to G_{i+1}$, and all other F_{ij}'s are empty.

In case (A), when the families are all empty, we have the free product of the groups G_i, since the universal property is precisely the (categorical) definition of the free product.

In case (B), we have $G = \mathbf{Z}_{n-1}$, since the point 1 is identified with n (unless $n = 1$, in which case we have $\mathbf{Z}$). If G' is any element of our category, then there is a unique morphism $Z_{n-1} \to G'$ sending 1 to $\phi'(1)$. This is well-defined since $\phi'(n) = \phi'(1)$, so $n - 1$ is in the kernel of ϕ', so the map $\phi' : \mathbf{Z} \to G'$ factors through $\mathbf{Z}_{n-1}$.

(C) is the usual definition of the amalgamated free product $G_1 *_{G_3} G_2$.

(D) is the direct limit of the groups G_i, as defined in the last homework.

Because of example (D), some authors (like Serre in his book on trees) call the universal object of our category a direct limit no matter what the groups G_i or the families F_{ij} are.