Algebraic Topology
Homework 1: Due September 1

There will be weekly homework assignments due each Wednesday at the *beginning* of class. You may study in teams, but it's a good idea to try the problems on your own first. Then discuss them and work together with classmates, friends, pets, favorite oracle, etc. In the end, I expect you to write up your own solutions to the problems.

These homework assignments will have a large overlap with the homework that I assigned last fall, as there are only so many good problems in the book! For that reason, I have taken down the homework solutions from last year's class web page. They'll go back up in December, in time to help you study for prelims. I hope to post solutions to this year's problems each week, shortly after they are due.

Most students will write out their solutions by hand (legibly, please!), but some may prefer to TeX things. Either way is fine.

Problem 1. Find an example of a topological space X that is not Haussdorf such that each point in X has a neighborhood homeomorphic to the open interval $(-1, 1)$.

Problem 2. Show that the figure 8 (viewed as a subset of the plane, with a topology induced from the usual topology of $\mathbb{R}^2$) is not homeomorphic to a circle.

Problem 3. Let T_1 be the surface of revolution obtained by rotating the circle $(x-2)^2 + y^2 = 1$ around the y axis. Let T_2 by the closed unit square with opposite edges identified. (Explicitly, $T_2 = [0, 1] \times [0, 1]/\sim$, where $(x, 0) \sim (x, 1)$ and $(0, y) \sim (1, y)$ for all $0 \leq x \leq 1$ and all $0 \leq y \leq 1$.) Give T_1 the topology inherited from $\mathbb{R}^3$ and give T_2 the quotient topology. Prove that T_1 and T_2 are homeomorphic.

The point of this exercise isn't to see that the torus is the same thing as a square with opposite sides identified – it's to understand the quotient topology well enough to prove it.

Problem 4. Massey, page 13, exercise 5.1

Problem 5 Showing that Euclidean spaces aren't homeomorphic can be remarkably difficult in higher dimensions. It's easy to see that $\mathbb{R}^0$ and $\mathbb{R}^1$ are not homeomorphic (prove it!), and that $\mathbb{R}^1$ and $\mathbb{R}^2$ are not homeomorphic (prove that, too!). But what about $\mathbb{R}^2$ and $\mathbb{R}^3$? These "obviously" are different, but can you prove it? Discuss how you would attempt such a proof. You probably don't have the tools to make a rigorous proof yet, but I'd like you to write down a few ways in which the spaces differ, and discuss the challenges of showing that these differences are preserved by homeomorphism. (Eventually we will develop tools, such as local homology and relative homology, that distinguish between $\mathbb{R}^n$ and $\mathbb{R}^m$ for all $n \neq m$.)