Algebraic Topology

Homework 10: Due Wednesday, November 10

Problem 1. As we defined in class, a short exact sequence $0 \to A \to B \to C \to 0$, with maps $i : A \to B$ and $j : B \to C$ *splits* if there is an isomorphism $\alpha : B \to A \oplus C$ with $\alpha \circ i$ being inclusion in the first factor and $j \circ \alpha^{-1}$ being projection onto the second. Prove that the sequence splits if (and only if) there exists a homomorphism $\ell : B \to A$ such that $\ell \circ i$ is the identity on A.

Problem 2. Prove the 5-lemma. Yes, we did this in class, but writing out the details of this sort of diagram-chasing will give you a better feel for the technique. Of course you can just copy the notes you (possibly) took in class, and you're free to do so, but please try to prove it on your own first.

Problem 3. Likewise, prove the snake lemma. You don't need to construct the maps in the long exact sequence, but you do need to show that the sequence is exact. I'll do most, if not all, of this in class, but the only way to get a feel for the ∂_* operator is to do a bunch of calculations and proofs with it.

The last three problems are exercises in using the Mayer-Vietoris sequence to compute homology. In each case you can assume that the preliminary version of Mayer-Vietoris is true, even though we haven't proven it yet. (I expect to cover this preliminary version on November 3 or 5.)

Problem 4. Compute the homology of $\mathbb{R}^n$ with p points removed. [Hint: use induction on p.]

Problem 5. Compute the homology of $\mathbb{R}^3$ with p non-intersecting lines removed. Note that the lines may not be parallel, so this doesn't instantly reduce to $\mathbb{R}^2$ with p points removed. (Yes, if you're clever you can find a way to show that a plane with p points removed is a deformation retract, but that probably requires techniques from the second semester prelim class. Tackling the problem directly, in analogy to what you did in problem 4, is easier.)

Problem 6. Now consider $\mathbb{R}^3$ with n_0 points removed and n_1 lines removed, with none of the lines intersecting, and none of the points on any of the lines. What is the homology of this space? [Hint: Mayer-Vietoris doesn't just compute the homology of $U \cup V$ from that of U, V and $U \cap V$, but it can also be used to compute the homology of $U \cap V$ from that of U, V and $U \cup V$.]