Algebraic Topology

Homework 12: Due Wednesday, November 24

(If you're going to miss Wednesday's class because of Thanksgiving, please drop off your homework before you go.)

Let $X_1, X_2, \ldots$ be a sequence of path-connected topological spaces, and suppose that $x_i \in X_i$ for each i. The *wedge* of the X's, denoted $\vee_i X_i$, is the quotient space of the disjoint union $\coprod_i X_i$ by the identification $x_i \sim x_j$ for all i, j. For instance, the wedge of two circles is a figure 8. The topology of $\vee_i X_i$ is that a set is open if (and only if) its intersection with each X_i is open. For the wedge of a finite number of spaces, this definition is obvious, but it is more subtle for infinite wedges. (The Hawaiian earring is *not* an infinite wedge of circles!)

Problem 1 Prove the following: If $Y = \vee_i X_i$ for a collection of spaces X_i, and if each x_i has a neighborhood that (strongly) deformation retracts to x_i, then $\tilde{H}_k(Y) = \oplus_i \tilde{H}_k(X_i)$. [Warning: as with Van Kampen's theorem, which says that the fundamental group of Y is the free product of the fundamental groups of the X_i's, the condition of having an appropriate neighborhood is crucial. For finite collections, this problem could be solved by induction and Mayer-Vietoris, but there is a better way. For infinite collections, don't even think about induction.]

Problem 2. Suppose that the topological spaces X and Y are path-connected. Show that $H_1(X \times Y)$ is isomorphic to $H_1(X) \oplus H_1(Y)$. [More generally, there is a simple formula for the free part of $H_n(X \times Y)$ in terms of the homology of X and Y, but the torsion part is more complicated. If H_k^f denotes the free part of H_k, then $H_n^f(X \times Y) = \oplus_k (H_k^f(X) \otimes H_{n-k}^f(Y))$. No, you are *not* expected to prove this!]

Since $H_n(S^n)$ is infinite cyclic, there are two choices of generators for $H_n(S^n)$. Each is called an *orientiation* of S^n. Given a choice of orientation, we can think of the top homology as the integers. If $f : S^n \to S^n$ is a continuous map between oriented spheres, then $f_* : \mathbf{Z} = H_n(S^n) \to \mathbf{Z} = H_n(S^n)$ is multiplication by an integer, which we call the *degree* of f. Note that, for maps from an n-sphere to itself, the degree doesn't depend on the orientation (since switching the orientation would involve multiplying by -1 twice), but for maps between distinct spheres, you have to pick orientations before you can define a degree. Note that $deg(f \circ g) = deg(f)deg(g)$ if both f and g are maps $S^n \to S^n$, and that $deg(f_0) = deg(f_1)$ if f_0 and f_1 are homotopic.

Problem 3. Consider the reflection map $R(x_1, x_2) = (-x_1, x_2)$ that maps $\mathbb{R}^2$ to itself, and also maps S^1 to itself. Show that R has degree -1. [Hint: Pick an explicit generator α of $H_1(S^1)$ such that $R_\#(\alpha) = -\alpha$.] Now consider reflection on the n-sphere, where $R(x_1, x_2, \ldots, x_{n+1}) = (-x_1, x_2, \ldots, x_{n+1})$. Show that R has degree -1.

Problem 4. Show that the antipodal map of S^n is homotopic to the identity if, and only if, n is odd.

The van Kampen computation of π_1 of a surface, when Abelianized, essentially gives the Mayer-Vietoris computation of H_1 of the surface. The following problem generalizes that idea.

Problem 5. Let U, V, and $X = U \cup V$ be as in the setup for Van Kampen's theorem. That is, each is open and path-connected and contains the base point x_0. Without using homology, write down (and prove) an "Abelianized van Kampen's theorem" that relates $\pi^A(X)$ to $\pi^A(U)$, $\pi^A(V)$, $\pi^A(U \cap V)$ and the way these groups map to one another, where π^A denotes the Abelianization of π_1. (You can take the standard van Kampen's theorem as a given – there's no need to reprove that!) Then write down the (reduced) Mayer-Vietoris sequence that computes $H_1(X)$ from $H_1(U)$, $H_1(V)$ and $H_1(U \cap V)$, and show how the maps in the M-V sequence relate to the maps in your Abelianized van Kampen's theorem.

Book problems: Page 191, problem 2.5 and page 201, problem 3.7.