Algebraic Topology

Homework 13: Due Friday, December 3

Problem 1. We saw in class how RP^2 is a CW complex with one 2-cell, one 1-cell, and one 0-cell, with $\partial e^{(2)} = 2e^{(1)}$ and $\partial e^{(1)} = 0$, which implies that $H_2(RP^2) = 0$, $H_1 = \mathbf{Z}_2$ and $H_0 = \mathbf{Z}$. Find a CW decomposition for $RP^2 \times RP^2$ and use it to compute the homology of $RP^2 \times RP^2$. This example provides a counterexample to an obvious guess of how the homology of $X \times Y$ is related to the homology of X and Y. The obvious formula works for the free part of the homology, but the torsion part behaves differently.

Problem 2. Let X be the torus of revolution $(\sqrt{x^2+z^2} - 2)^2 + y^2 = 1$. Let Y be the union of X with the two disks $(x-2)^2 + y^2 \leq 1, z = 0$ and $(x+2)^2 + y^2 \leq 1, z = 0$. Exhibit a CW structure for this torus-with-two-disks space (which appeared on last year's midterm), and use this structure to compute the homology of this space.

Problem 3. Construct a 3-dimensional CW complex (that is, at least one 3-cell and no cells of dimension greater than 3) with the following homology groups: $H_0 = \mathbf{Z}$, $H_1 = \mathbf{Z}_2 \oplus \mathbf{Z}_3$, $H_2 = \mathbf{Z}$, $H_3 = 0$. There are many, many possible answers.

Problem 4. Let X be a simplicial complex with a finite number of simplices (aka a "finite simplicial complex"). Prove that the simplicial homology of X is isomorphic to the singular homology of X. [Hint: writing X as a CW complex is easy, and this gives a correspondence between simplicial chains and cellular chains. What takes work is showing that the boundary maps in the simplicial complex are the same as the maps in the cellular computation of the singular homology.]

Let G_1, $G_2, \ldots$ be Abelian groups, and suppose that we have maps $\rho_n : G_n \to G_{n+1}$. The *direct limit* of the groups G_n is the quotient of the product $G_1 \times G_2 \times \cdots$ by the identification $x \sim y$ if $x \in G_n$, $y \in G_{n+1}$, and $y = \rho_n(x)$. That is, G_1 is identified with a subgroup of G_2, G_2 is identified with a subgroup of G_3, etc. The groups G_n are called *approximants* to the direct limit.

The direct limit of spaces is similar. Let $X_1, \ldots,$ be spaces, and consider continuous maps $f_n : X_n \to X_{n+1}$. As a set, the direct limit of the X_n's is the union $\coprod X_n$, modulo the identifications $x_n \sim f_n(x_n)$ if $x_n \in X_n$. A set is considered closed if its intersection with (the image of) each X_n is closed (in X_n).

The most common example is when $X_1 \subset X_2 \subset \cdots$, so the maps f_n are just inclusions. The direct limit is then the union, albeit with a particular topology. For instance, X_n might be the n-skeleton of a CW complex, or X_n might be a finite sub-complex of an infinite (but finite-dimensional) CW complex.

CW complexes have the following useful property that you are free to use below: Every compact subspace is contained in a finite sub-complex, and in particular in a finite skeleton X_n. A proof may be found on p520 of Hatcher, but here's a sketch. If a compact set C hits infinitely many cells, then we could find a subset $S = \{x_1, x_2, \ldots\}$ of C with each x_i in a different cell. Since S hits the interior of each cell at most once, it hits the closure of each cell only a finite number of times (I'm skipping some details here), so the intersection of S with each cell is closed, so S is closed. Being a closed subset of the compact C, S

must be compact. But S has the discrete topology, and so is not compact. Contradiction.

Problem 5. Let X be the direct limit of a collection of spaces $X_1 \subset X_2 \subset \cdots$, such that any compact subset of X lies in some X_n. Prove that the homology of X is the direct limit of the homology of X_n.

Problem 6. As a corollary to Problem 5, prove the following theorem from last week's lectures: Let X be a CW complex, and let F^n be a free abelian group whose generators are the n-cells of X. Then there exist maps $\partial_n : F^n \to F^{n-1}$ with $\partial^2 = 0$ such that the n-th singular homology of X is isomorphic to $ker(\partial_n)/Im(\partial_{n+1})$. (We can take $\partial_0 : F^0 \to 0$ to be the zero map.) When X is finite-dimensional, this was proved in class, and you are free to use those results without reproving them. The trick is extending this to infinite dimensions.

Problem 7. The inclusion $\mathbb{R}^n \to \mathbb{R}^{n+1}$, where $(x_1, \ldots, x_n)$ goes to $(x_1, \ldots, x_n, 0)$, induces an inclusion of S^{n-1} into S^n. Let S^∞ be the direct limit of $S^1 \subset S^2 \subset \cdots$, and let RP^∞ be the quotient of S^∞ by the antipodal map. Let CP^∞ be the direct limit of the spaces CP^n induced by the obvious inclusion of each C^n in C^{n+1}. Compute the homologies of S^∞, RP^∞, and CP^∞.