Algebraic Topology
Homework 4: Due September 22

1. Page 53, problem 7.3. Note that "infinite product" means "with the product topology". Except where specifically noted, infinite products always have the product topology.

2. To see how evil alternatives to the product topology are, consider $X = S^1 \times S^1 \times \cdots$ with the *box* topology, where for definiteness we're taking S^1 to be the unit circle in the complex plane and taking the index set to be the natural numbers. (In the box topology, the set $U_1 \times U_2 \times \cdots$ is open if each U_i is open. Those infinite boxes form a base for the topology.) Give a characterization of all the continuous loops in X, and use this characterization to compute $\pi_1(X, x)$, where $x = (1, 1, 1, \ldots)$.

3. Let $Z[1/2]$ denote the set of rational numbers whose denominators are powers of 2. (This includes integers, since $1 = 2^0$.) Show that $Z[1/2]$ is an Abelian group under addition. Then show, in two ways, that $Z[1/2]$ is not finitely generated: First show this directly, by showing that the subgroup generated by any finite set of elements of $Z[1/2]$ cannot be the entire group. Second, show that $Z[1/2]$ does not satisfy the conclusions of Theorem 3.6 on page 70 (the classification theorem for finitely generated Abelian groups).

4. The classification theorem for finitely generated Abelian groups is stated in different ways in different books. One formulation says that each such group can be written as $Z^n \oplus Z_{a_1} \oplus \cdots \oplus Z_{a_k}$, where each a_i is a (positive) power of a prime, and that the numbers a_i are unique, up to permutation. Another formulation says that the group can be written as $Z^n \oplus Z_{t_1} \oplus \cdots \oplus Z_{t_\ell}$, where each t_i divides t_{i+1}, and that the numbers t_i are unique. Give an algorithm for converting from each of these canonical forms to the other. Then use this algorithm to show that each formulation of the theorem implies the other. (Note: you do not have to prove either formulation from scratch. Just show that each implies the other.)

5. Let C be a category. Two objects X, Y in C are called "isomorphic" if there are morphisms $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ is the identity morphism on Y and $g \circ f$ is the identity on X. An object Z in C is called "initial" or "universal" if, for every object Y in C, there is a unique morphism $Z \to Y$. For instance, in the category of groups and group homomorphisms, the trivial group is universal. In the category of sets and functions, the empty set is universal.

Show that any two universal objects are isomorphic, and show that any object that's isomorphic to a universal object is also universal. In other words, the universal object is "unique up to isomorphism".

[The dual of a universal object is a "terminal" or "final" object. An object T is final if, for each object Y, there is a unique morphism $Y \to T$. The proof you just gave, with arrows reversed, shows that final objects are unique up to isomorphism.

To every category C there is an *opposite category* C^{op}, such that the objects of C^{op} are the same as the objects of C, and such that $Hom_{C^{op}}(X, Y) = Hom_C(Y, X)$. In other words, the morphisms are the same as in C, only with all the arrows reversed. Compositions are defined in the opposite order. Universal objects in C become final objects in C^{op}, and vice-versa.]

6. Find examples of categories with each of the following properties: (1) a universal object but no final object. (2) no universal object and no final object, (3) a universal object and a final object, with those objects not isomorphic, and (4) universal and final objects that are the same. (Objects that are both universal and final are called "zero objects")

7. Let X be a topological space. We build a category out of X whose objects are the open sets in X. If U and V are open sets, then $Hom(U, V)$ is empty unless $U \subset V$, in which case it consists of the inclusion map of U into V. In this category, what are the universal object(s) and final object(s)? (Justify your answers, of course.)