Algebraic Topology

Homework 5: Due Friday, October 1

This homework is all about reduced words, free products, and variations on free products. Note that it's fairly long, and also that it's due on Friday, not Wednesday – solutions will be posted when I return from Korea. Subsequent homeworks will be due on Friday, October 8 and Wednesday October 20, since we have a midterm on October 15.

1. Page 74, problems 4.1, 4.3, 4.4, 4.9.

2. Page 77, problem 5.3.

3. Page 81, problem 6.1

4. In class, I sketched an argument for the claim that the set of all reduced words is a group. Flesh out this argument by defining the product of two arbitrary reduced words and showing that this product is associative.

5. Suppose we have a group G and several elements $\{g_i\}$ in G. Show that there exists a normal subgroup K of G containing all the g_i's, such that K is contained in every normal subgroup that contains the g_i's. This is called the *normal subgroup generated by the g_i's*. Suppose that H is another group and $f : G \to H$ is a homomorphism. Let $p : G \to G/K$ be the obvious projection. Show that f lifts to a map $\hat{f} : G/K \to H$ if and only if every $f(g_i)$ is the identity. (By "lift to a map" I mean that $\hat{f}$ exists such that $f = \hat{f} \circ p$.)

6. Let G_1 and G_2 be two groups, and let H inject in both of them via injections i_1 and i_2. The *amalgamated free product of G_1 and G_2 over H*, denoted $G_1 *_H G_2$, is the quotient of $G_1 * G_2$ by the normal subgroup generated by $\{i_1(h)i_2(h)^{-1}\}$, where h ranges over H. It's like the free product of G_1 and G_2, only with $i_1(H)$ identified with $i_2(H)$. Let $\phi_1 : G_1 \to G_1 *_H G_2$ and $\phi_2 : G_2 \to G_1 *_H G_2$ be the obvious injections.

Suppose we have a group B and maps $\psi_1 : G_1 \to B$ and $\psi_2 : G_2 \to B$ such that $\psi_1 \circ i_1 = \psi_2 \circ i_2$. Show that there exists a unique homomorphism $f : G_1 *_H G_2 \to B$ such that $\psi_1 = f \circ \phi_1$ and $\psi_2 = f \circ \phi_2$.

7. Define a relevant category for which $G_1 *_H G_2$ is the universal object. In other words, express the conclusion of problem 6 as a universal property.

8. Repeat problems 6 and 7, only with i_1 and i_2 no longer assumed to be injective. For this problem, they're just group homomorphisms. Note that ϕ_1 and ϕ_2 are no longer necessarily injective. As far as I know, there is no standard term for the thing that replaces $G_1 *_H G_2$ – let's call it the generalized amalgamated product and denote it $G_1 \tilde{*}_H G_2$. (Some authors do use "amalgamated free product" to mean "generalized amalgamated free product", and they denote it $G_1 *_H G_2$, but others reserve the term for the case where H is a subgroup of G_1 and also a subgroup of G_2.)

The construction of Problem 8 is extremely important, thanks to van Kampen's theorem (aka the Seifert-van Kampen theorem), which says that the the fundamental group of the union of two open sets U and V is the generalized amalgamated free product $\pi_1(U, x_0)$ and $\pi_1(V, x_0)$ over $\pi_1(U \cap V, x_0)$. where $U \cap V$ is assumed path-connected, $x_0 \in U \cap V$, and the maps i_1 and i_2 are induced from the inclusions $U \cap V \to U$ and $U \cap V \to V$.

We're going to spend a lot of time trying to understand the topology of this in Chapter 4. Chapter 3 is all about setting up the necessary algebra.

9. A free abelian group on 3 generators cannot inject in a free abelian group on two generators, but non-Abelian free groups are different. Let F_2 be the free group on two generators a, b, and let F_3 be the free group on generators s_1, s_2, s_3. Consider the map $f : F_3 \to F_2$ defined by $f(s_1) = ab$, $f(s_2) = a^2b^2$, $f(s_3) = a^3b^3$. Show that f is an injection.

10. Generalize the construction of problem 9 to construct a subgroup of F_2 that is not finitely generated. (You can take as given the fact that a free group on infinitely many generators is not finitely generated.)