Algebraic Topology

Homework 6: Due October 8

1. Page 94, problems 3.1 and 3.2. Both of these were essentially done in class, but spell them out anyway.

2. As noted in class, the *Hawaiian Earring* is the union of infinitely many circles of decreasing diameter, all meeting at a point. For instance, for each positive integer n let S_n be a circle of radius $1/n$ in the plane, centered at $(1/n, 0)$, and let $X = \cup_n S_n$. (It helps to DRAW A PICTURE!) Let $x_0 = (0,0)$. Show that this example does not meet the hypotheses of Exercise 3.2. (Again, already more or less done in class.)

3. Not only does the Hawaiian Earring not meet the hypotheses of Exercise 3.2, it also doesn't meet the conclusions. Show that the inclusions $i_n : S_n \hookrightarrow X$ induce injections $\pi_1(S_n, x_0) \to \pi_1(X, x_0)$, and hence an injection $f : \prod^* \pi_1(S_n, x_0) \hookrightarrow \pi_1(X, x_0)$, but that f is not onto. In other words, $\pi_1(X, x_0)$ is *not* the free product of all of the cyclic groups $\pi_1(S_n, x_0)$. [Hint: Can every loop in X be expressed by a finite concatenation of loops in the S_n's?]

4. Page 94, problem 3.4. Hint: with the correct choice of open sets, this is an easy corollary of problem 3.2.

5. Page 95, problem 3.7.

5. Page 103, problem 5.4.

6. Let X_2 be a path-connected Hausdorff space, let $k > 2$, and let $f : S^{k-1} \to X_2$ be a continuous map. Let B be the closed unit ball in $\mathbb{R}^k$, and for each $x \in S^{k-1} \subset B$, identify x with $f(x) \in X_2$. Let X be the union of X_2 with B, modulo these identifications. This is called *adjoining a k-cell to X_2*. Let $x_0 \in X_2$. Show that $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X_2, x_0)$.

7. Let X_1 be another path-connected Hausdorff space, and let X_2 be obtained by adjoining a 2-cell to X_1. Let $x_0 \in X_1$. Show that the inclusion $X_1 \hookrightarrow X_2$ induces a surjection $\pi_1(X_1, x_0) \to \pi_1(X_2, x_0)$. Give an example where this surjection is not an isomorphism.

The importance of exercises 6 and 7 is that it is possible to build any manifold, and a lot of other spaces, recursively. To get a space X, start with a set X_0 of disconnected points, called the 0-skeleton of X. Then add 1-cells to get a graph X_1 that is called the 1-skeleton. Then add 2-cells to X_1 to get X_2, add 3-cells to X_2 to get X_3, and in general add k-cells to the $k-1$ skeleton X_{k-1} to get the k-skeleton X_k. Finally, take $X = \cup_k X_k$. The upshot is that $\pi_1(X)$ is isomorphic to $\pi_1(X_2)$, and is a quotient of $\pi_1(X_1)$ by some relations that come from the 2-cells. This is both a blessing and a curse. The blessing is that we don't have to worry about higher dimensional structures when computing a fundamental group. The curse is that we can't use the fundamental group to keep track of higher dimensional structures. To do that we need other tools, such as homology.