Algebraic Topology

Homework 8: Due Wednesday, October 27

Most of this week's problem set is about simplicial homology, first of triangulated surfaces, and then of simplicial complexes in general. But first some leftover problems on covering spaces:

Problem 1 Page 135, problem 7.2 – For example 2.4, you can restrict attention to the cover $\mathbb{R}^2 \to T^2$.

Problem 2 Page 144, problem 10.1

Given a triangulated surface, we can order the vertices and describe edges and triangles by their vertices. If a triangle has three vertices v_i, v_j and v_k, and if $i < j < k$, we denote the triangle they span by t_{ijk}. Likewise e_{ij}, with $i < j$, is the edge from v_i to v_j. Let C_0 be the free abelian group whose generators are the vertices of the triangulation, let C_1 be the free abelian group whose generators are the edges, and let C_2 be the free abelian group whose generators are the triangles. In other words, a 2-chain is a formal sum of triangles, a 1-chain is a formal sum of edges, and a 0-chain is a formal sum of vertices. We define $\partial_2 t_{ijk} = e_{jk} - e_{ik} + e_{ij}$ and $\partial_1 e_{ij} = v_j - v_i$ and extend by linearity to all chains. Let C_k be zero for $k < 0$ or $k > 2$, and let $\partial_k : C_k \to C_{k-1}$ be the zero map when k is anything other than 1 or 2. It's easy to check that $\partial_k \circ \partial_{k+1} = 0$ for all k, so the kernel of ∂_k contains the image of ∂_{k+1}, and we can define the *kth simplicial homology* of the triangulated surface to be the kernel of ∂_k modulo the image of ∂_{k+1}.

Problem 3 Revisit problems 7.2, 7.3, and 7.4 on page 19, only now compute the simplicial homology of each triangulation.

We now generalize to higher dimensions. Working in $\mathbb{R}^k$, the *standard k-simplex* is the convex hull of the points $v_0 = (0, 0, \ldots, 0)$, $v_1 = (1, 0, 0, \ldots, 0)$, $v_2 = (0, 1, 0, \ldots, 0)$, through $v_k = (0, 0, \ldots, 0, 1)$. We will label this is Δ^k_0. Now let A be a rank-k affine map from $\mathbb{R}^k$ to $\mathbb{R}^n$ that takes $v_0, \ldots, v_k$ to $k + 1$ distinct points $w_0, \ldots, w_k$. The image of Δ^k_0 under this map is called a k-simplex with vertices $w_0, \ldots, w_k$ and is denoted $\Delta_{w_0,\ldots,w_k}$. We order the vertices in any simplicial complex, and generally list the vertices of each simplex in increasing order.

(Actually, it's sometimes useful to write the vertices in arbitrary order, with the understanding that t_{jik} means $-t_{ijk}$, etc. That is, permuting the order of the vertices means multiplying by the sign of the permutation. For instance, $\partial t_{ijk} = e_{jk} - e_{ik} + e_{ij}$, regardless of the ordering of i, j, and k. Still, we usually take as a basis for C^k the k-simplices with vertices written in increasing order.)

The boundary of a simplex $\Delta_{w_0,\ldots,w_k}$, is defined to be $\sum_{i=0}^{k} (-1)^i \Delta_{w_0,\ldots,w_{i-1},w_{i+1},\ldots w_k}$. The $k-1$ simplex based on all the vertices except w_i is called the i-th *face* of the k-simplex.

Problem 4. Show that $\partial_{k-1} \circ \partial_k = 0$. Also show that the definition of $\partial\Delta_{w_0,\ldots,w_k}$ works even if the vertices are NOT listed in increasing order.

A *simplicial complex* is a collection of simplices such that (1) every collection of vertices determines at most one simplex (for instance, there can't be two edges from v_i to v_j), and

(2) the intersection of any two simplices is the simplex defined by their common vertices (or is empty if they have no vertices in common), and (3) if a simplex is in the collection, then so are all its faces, and the faces of the faces, etc. (e.g., if t_{ijk} is a 2-simplex in the collection, then e_{ij}, e_{ik} and e_{jk} are all 1-simplices in the collection, and v_i, v_j and v_k are all 0-simplices.) The simplest simplicial complex of dimension k consists of a single k-simplex, together with its faces and sub-faces.

Problem 5. Compute the homology of the 2-simplex Δ_0^2 (together with its faces and sub-faces, of course). Repeat for Δ_0^3. Repeat for the complex obtained from the faces of Δ_0^3 without the 3-simplex itself (i.e., the surface of a tetrahedron).