Algebraic Topology Solutions
Homework 1: Due September 1

Problem 1. Find an example of a topological space X that is not Haussdorf such that each point in X has a neighborhood homeomorphic to the open interval $(-1, 1)$.

Consider the space $(\{1, 2\} \times \mathbb{R})/\sim$, where $(1, x) \sim (2, x)$ for all $x \neq 0$. This is the real line with the origin doubled, with a funny topology at the origin. Remember that a set is open in the quotient if and only if its preimage is open. Suppose $a < 0 < b$. Since $\{1\} \times (a, b) \cup \{2\} \times (a, 0) \cup \{2\} \times (0, b)$ is open in $\{1, 2\} \times \mathbb{R}$, the interval from a to b including one copy of the origin but not the other is open in the quotient. This shows that each copy of the origin has a neighborhood homeomorphic to an open interval. (All other points manifestly have nice neighborhoods – just take an interval that doesn't reach the origin.)

Problem 2. Show that the figure 8 (viewed as a subset of the plane, with a topology induced from the usual topology of $\mathbb{R}^2$) is not homeomorphic to a circle.

Let X be the figure 8, and let $p \in X$ be the crossing point. Suppose that $f : X \to S^1$ is a homeomorphism. Then f is also a homeomorphism from $X - \{p\}$ to $S^1 - \{f(p)\}$. But the first set is disconnected while the second is not, so we have a contradiction.

Problem 3. Let T_1 be the surface of revolution obtained by rotating the circle $(x-2)^2 + y^2 = 1$ around the y axis. Let T_2 by the closed unit square with opposite edges identified. (Explicitly, $T_2 = [0, 1] \times [0, 1]/\sim$, where $(x, 0) \sim (x, 1)$ and $(0, y) \sim (1, y)$ for all $0 \leq x \leq 1$ and all $0 \leq y \leq 1$.) Give T_1 the topology inherited from $\mathbb{R}^3$ and give T_2 the quotient topology. Prove that T_1 and T_2 are homeomorphic.

Another formula for T_1 is $(\sqrt{x^2 + z^2} - 2)^2 + y^2 = 1$. Let $f : [0, 1]^2 \to \mathbb{R}^3$ be given by $f(u, v) = ((2 + \cos(2\pi u)) \cos(2\pi v), \sin(2\pi u), (2 + \cos(2\pi u)) \sin(2\pi v))$. Since $f(0, v) = f(1, v)$ and $f(u, 0) = f(u, 1)$, f induces a map from T_2 to $\mathbb{R}^3$. I claim that the image of f is T_1.

A useful fact is that a continuous bijection $f : X \to Y$ between compact Hausdorff spaces always has a continuous inverse. This is because the continuity of f^{-1} is the same as saying that $f(U)$ is open for every open $U \subset X$. If U is open, then U^c is closed, hence compact, so $f(U^c)$ is compact, hence closed, so $f(U) = f(U^c)^c$ is open.

It's easy to check that $f(u, v)$ satisfies the equation for T_1, and that f is continuous. We must show that f, viewed as a map from T_2 to T_1, is 1–1 and onto. If $f(u, v) = f(u', v')$, then $y = \sin(2\pi u) = \sin(2\pi u')$ and $x^2 + z^2 = (2 + \cos(2\pi u))^2 = (2 + \cos(2\pi u')^2$, implying that $\cos(2\pi u) = \cos(2\pi u')$, hence that u and u' differ by an integer. By looking at $x/\sqrt{x^2 + z^2}$ and $z/\sqrt{x^2 + z^2}$ we see that v and v' differ by an integer. In other words, distinct points in T_2 map to distinct points in T_1.

To see that every point in T_1 is in the image of f, we know that $(\sqrt{x^2 + z^2} - 2)^2 + y^2 = 1$, so there exists an angle $0 \leq \theta < 2\pi$ with $\cos(\theta) = \sqrt{x^2 + z^2} - 2$ and $\sin(\theta) = y$. Since $|\sqrt{x^2 + z^2} - 2| \leq 1$, $x^2 + z^2 \neq 0$, so we can also find ϕ such that $\cos(\phi) = x/\sqrt{x^2 + z^2}$ and $\sin(\phi) = z/\sqrt{x^2 + z^2}$. Then take $u = \theta/2\pi$ and $v = \phi/2\pi$ and observe that $f(u, v) = (x, y, z)$.

Problem 4. Massey, page 13, exercise 5.1

Compactness is trivial, since we are taking the quotient of a compact set (a closed polygon) by some identifications. The quotient being Haussdorff is done last.

What's interesting is showing that the quotient of a $2n$-gon by identifying edges in pairs is always a surface. That is, that every point has a neighborhood homeomorphic to an open ball in $\mathbb{R}^2$.

A point in the quotient space either corresponds to one point p in the interior of the polygon, to two points q_1, q_2 on identified edges (but not vertices) of the polygon, or several vertices $r_1, \ldots, r_k$ of the polygon.

In the first instance, we can take our open set to be a ball around p_1 that doesn't reach the boundary of the polygon. The identification don't come into it, and this ball is homeomorphic to a ball in $\mathbb{R}^2$.

In the second instance, a neighborhood can be taken to be a half-disk around q_1 and a half-disk around q_2, joined on their common diameter to make a whole disk.

In the third instance, if $k = 1$ then a neighborhood of r_1 is a pie wedge with edges identified, i.e. a cone, which is homeomorphic to a disk.

If $k > 1$, we consider the edge parts that run into or out of the various vertices r_i. (E.g., front of edge a, back of edge a, back of edge b, etc.) Each part occurs exactly twice in the polygon, and hence exactly twice in the list. Call these edge parts $e_1, \ldots, e_k$, and label them (and renumber the vertices) so that r_1 is connected to edge parts e_1 and e_2, r_2 is connected to e_2 and e_3, ..., e_{k-1} is connected to e_{k-1} and e_k, and r_k is connected to e_k and e_1. [Note: if we ever came back to e_1 without seeing all of the other edges twice, then we would have grouped the vertices r_i into two classes with distinct sets of edges, which would therefore NOT be identified in the quotient space.]

A neighborhood of our point in the quotient corresponds to a bunch of pie wedges, sandwiched between edges e_1 and e_2, e_2 and e_3, etc. There is no guarantee that the geometry is nice, as the sum of the interior angles of these pie wedges doesn't have to add up to 2π, but each wedge is homeomorphic to a pie wedge of angle $2\pi/k$, so their union is homeomorphic to an entire pie (i.e., to an open ball in $\mathbb{R}^2$.)

Finally, we need to check that our space is Hausdorff. Given two distinct points, it's easy to see that the disks we've just constructed can be taken small enough to not overlap.

Problem 5 Showing that Euclidean spaces aren't homeomorphic can be remarkably difficult in higher dimensions. It's easy to see that $\mathbb{R}^0$ and $\mathbb{R}^1$ are not homeomorphic (prove it!), and that $\mathbb{R}^1$ and $\mathbb{R}^2$ are not homeomorphic (prove that, too!). But what about $\mathbb{R}^2$ and $\mathbb{R}^3$? These "obviously" are different, but can you prove it? Discuss how you would attempt such a proof. You probably don't have the tools to make a rigorous proof yet, but I'd like you to write down a few ways in which the spaces differ, and discuss the challenges of showing that these differences are preserved by homeomorphism. (Eventually we will develop a tool, called local homology, that distinguishes between $\mathbb{R}^n$ and $\mathbb{R}^m$ for all $n \neq m$.)

$\mathbb{R}^0$ is a single point, while $\mathbb{R}^1$ is infinite, so they can't be put in 1–1 correspondence, much less be homeomorphic.

$\mathbb{R}^1$ and $\mathbb{R}^2$ have the same cardinality, but removing a point from $\mathbb{R}^1$ yields a disconnected space, while removing a point from $\mathbb{R}^2$ doesn't. (See Problem 2 for the details of this argument.)

One approach for $\mathbb{R}^2$ versus $\mathbb{R}^3$ is to try to disconnect $\mathbb{R}^2$ with a line L. However, it's hard to say what the image of L will look like in $\mathbb{R}^3$. How do we *know* that it won't disconnect $\mathbb{R}^3$? I don't see much hope for this approach, but maybe it can be done with a clever trick that I haven't thought of.

Another approach is to show that $\mathbb{R}^3$ minus a point is simply connected, while $\mathbb{R}^2$ minus a point is not. At this point in the course we don't (officially) know what "simply connected" means, but once we've worked with the fundamental group we'll have the tools for making this argument rigorous.

The best tool I know for distinguishing spaces of different dimensions is homology. If $m < n$, then $\mathbb{R}^m$ minus a point has nontrivial H_{m-1}, while $\mathbb{R}^n$ minus a point has trivial H_{m-1}. The homotopy group π_{m-1} also distinguishes between $\mathbb{R}^m$ minus a point and $\mathbb{R}^n$ minus a point, and therefore between $\mathbb{R}^m$ and $\mathbb{R}^n$. The "local homology" that I mentioned in the statement of the problem also works, but is a little harder to describe than H_{m-1} or π_{m-1}.