Algebraic Topology

Solutions to homework 10: Due Wednesday, November 10

Problem 1. As we defined in class, a short exact sequence $0 \to A \to B \to C \to 0$, with maps $i : A \to B$ and $j : B \to C$ *splits* if there is an isomorphism $\alpha : B \to A \oplus C$ with $\alpha \circ i$ being inclusion in the first factor and $j \circ \alpha^{-1}$ being projection onto the second. Prove that the sequence splits if (and only if) there exists a homomorphism $\ell : B \to A$ such that $\ell \circ i$ is the identity on A.

If the sequence splits, then α followed by projection on the first factor works as ℓ.

For the converse, suppose that ℓ exists. Let $\alpha(b) = (\ell(b), j(b))$. If $\alpha(b) = 0$, then $j(b) = 0$, so $b = i(a)$. But $0 = \ell(b) = a$, so $b = i(0) = 0$. This shows that α is injective. For surjectivity, we must find, for each pair (a, c), a b such that $\alpha(b) = (a, c)$. Since j is surjective, there exists a b_0 such that $j(b_0) = c$. Let $b = i(a) + b_0 - i(\ell(b_0))$. Since $j \circ i = 0$, and since $\ell \circ i$ is the identity on A, $j(b) = j(b_0) = c$, and $\ell(b) = \ell \circ i(a) - \ell(b_0) - \ell(b_0) = a$.

Problem 2. Prove the 5-lemma. Yes, we did this in class, but writing out the details of this sort of diagram-chasing will give you a better feel for the technique. Of course you can just copy the notes you (possibly) took in class, and you're free to do so, but please try to prove it on your own first.

We are given an exact sequence, with maps i, j, k, and ℓ mapping A to B, B to C, C to D and D to E. We are also given another exact sequence with groups A', B', C', D' and E' and maps i', j', k', and ℓ'. Finally, we are given maps $\alpha : A \to A'$, $\beta : B \to B'$, $\gamma : C \to C'$, $\delta : D \to D'$ and $\epsilon : E \to E'$ such that all squares commute. We are told that α, β, δ and ϵ are isomorphisms, and must prove that γ is an isomorphism.

We did injectivity in class, but I'll repeat it. If $\gamma(c) = 0$, then $\delta \circ k(c) = k' \circ \gamma(c) = 0$. But δ is an isomorphism, so $k(c) = 0$, so c is in the image of j, and we write $c = j(b)$. Let $b' = \beta(b)$.

Since $0 = \gamma(c) = \gamma(j(b)) = j'(\beta(b)) = j'(b')$, b' is in the image of i', and we write $b' = i'(a')$. Since α is an isomorphism, $a' = \alpha(a)$ for a (unique) $a \in A$.

Now $b' = i'(\alpha(a)) = \beta(i(a))$. But $b' = \beta(b)$ and β is an isomorphism, so $b = i(a)$. But then $c = j(b) = j(i(a)) = 0$.

For surjectivity, pick $c' \in C'$ and let $d' = k'(c')$. This equals $\delta(d)$ for some (unique) $d \in D$, since δ is an isomorphism. Since $\epsilon \circ \ell(d) = \ell'(delta(d)) = \ell'(d') = \ell'(k'(c')) = 0$, we must have $ell(d) = 0$, so $d = k(c)$ for some $c \in C$. Note that $k'(c' - \gamma(c)) = 0$, so $c' - \gamma(c) = j'(b')$ for some b'. But $b' = \beta(b)$ for some b, and $j'(b') = j'(\beta(b)) = \gamma(j(b))$, so $c' = \gamma(c + j(b))$ is in the image of γ.

Problem 3. Likewise, prove the snake lemma. You don't need to construct the maps in the long exact sequence, but you do need to show that the sequence is exact. I'll do most, if not all, of this in class, but the only way to get a feel for the ∂_* operator is to do a bunch of calculations and proofs with it.

Recall the definition of the map ∂_*. If $[c] \in H_k(C)$, we find b such that $j(b) = c$, look at ∂b, find a such that $i(a) = \partial b$, and take $[a] = \partial_*[c]$. (This being well-defined involves showing that ∂b is in the image of i, that $\partial a = 0$, and that the class $[a]$ doesn't depend on the choices made, but we did all that in class.)

Exactness at A: If $[a] = \partial_*[c]$, then $i_*[a] = [i(a)] = [\partial b] = 0$, so the image of ∂_* is in the kernel of i_*. Now suppose that $i_*[a] = 0$. Then $i(a)$ is a boundary, so we write $a = \partial b$. Let $c = j(b)$. $\partial c = \partial j(b) = j(\partial b) = j(i(a)) = 0$, so $[c]$ is a well-defined homology class and $\partial_*[c] = [a]$. This shows that the kernel of i_* is in the image of ∂_*.

Exactness at B: $j_*(i_*[a]) = [j(i(a))] = 0$, so the image of i_* is in the kernel of j_*. Now suppose that $j_*[b] = 0$, where $b \in C_k(B)$. Then $j(b) = \partial c$ for some $c \in C_{k+1}(C)$. Pick $b' \in C_{k+1}(B)$ such that $j(b') = c$. (This is possible since j is surjective.) But then $j(b - \partial b') = j(b) - j(\partial b') = j(b) - \partial j(b') - j(b) - \partial(c) = 0$, so $b - \partial b' = i(a)$. Now $i(\partial a) = \partial(i(a)) = \partial(b - \partial b') = 0$, so $\partial a = 0$, and we have $[b] = [b - \partial b'] = [i(a)] = i_*[a]$.

Exactness at C. If a, b, c are as in the definition of ∂_*, then $i_*(\partial_*[c]) = [i(a)] = [\partial b] = 0$. Now suppose that $[a] \in H_k(A)$ and $i_*[a] = 0$. Then $i(a) = \partial b$ for some $b \in C_{k+1}(B)$ and we define $c = j(b)$. $\partial(c) = \partial(j(b)) = j(\partial(b)) = j(i(a)) = 0$, so $[c]$ is a homology class and $[a] = \partial_*[c]$. Ta da!

The last three problems are exercises in using the Mayer-Vietoris sequence to compute homology. In each case you can assume that the preliminary version of Mayer-Vietoris is true, even though we haven't proven it yet. (I expect to cover this preliminary version on November 3 or 5.)

Problem 4. Compute the homology of $\mathbb{R}^n$ with p points removed. [Hint: use induction on p.]

I'll give two solutions to show that $H_0 = \mathbf{Z}$ and $H_{n-1} = \mathbf{Z}^p$ and all other homology groups are trivial.

First solution: Work by induction on p. It's true for $n = 1$, since $\mathbb{R}^n$ with a point removed is homotopy equivalent to S^{n-1}. Take the points to be arranged at integer points $1, \ldots, p$ on the x_1 axis. Let $U = \{x_1 < p\}$ and let $V = \{x_1 > p-1\}$, so $U \cap V$ is contractible, so Mayer-Vietoris (using reduced homology) says that $\tilde{H}_k(U \cup V) = \tilde{H}_k(U) \oplus \tilde{H}_k(V)$. Since U is homeomorphic to $\mathbb{R}^n$ with $p-1$ points removed (i.e., something covered by the inductive hypothesis) and V is homeomorphic to $\mathbb{R}^n$ with one point removed, $H_k(U \cup V)$ is of the desired form.

Second solution: Let U be $\mathbb{R}^n$ with p points removed and let V be the union of p non-intersecting balls, centered at the p points. Since $U \cup V = \mathbb{R}^n$, Mayer-Vietoris tells us that $\tilde{H}_k(U \cap V) = \tilde{H}_k(U) \oplus \tilde{H}_k(V)$. But $U \cap V$ is the disjoint union of p punctured balls, each homotopy equivalent to S^{n-1}, and $\tilde{H}_k(V)$ is trivial for $k > 0$, so, for $k > 0$, $H_k(U) = H_k(U \cap V)$, which is $\mathbf{Z}^p$ if $k = n - 1$ and 0 otherwise. (We also have $H_0(U) = \mathbf{Z}$ since U is connected.)

Problem 5. Compute the homology of $\mathbb{R}^3$ with p non-intersecting lines removed. Note that the lines may not be parallel, so this doesn't instantly reduce to $\mathbb{R}^2$ with p points removed. (Yes, if you're clever you can find a way to show that a plane with p points removed is a deformation retract, but that probably requires techniques from the second semester prelim class. Tackling the problem directly, in analogy to what you did in problem 4, is easier.)

Let U be $\mathbb{R}^n$ with the p lines removed, let V be the union of neighborhoods of the p lines (which has the homotopy type of p points), so $U \cup V = \mathbb{R}^n$ and $U \cap V$ is the union of p punctured cylinders, each homotopy equivalent to a circle. As in the second solution to problem 4, for $k > 0$ we have $H_k(U) = H_k(U) \oplus H_k(V) = H_k(U \cap V)$, which is $\mathbf{Z}^p$ if $k = 1$ and is trivial if $k > 1$. Of course, $H_0(U) = \mathbf{Z}$.

One can also prove it by induction, removing one line at a time.

Problem 6. Now consider $\mathbb{R}^3$ with n_0 points removed and n_1 lines removed, with none of the lines intersecting, and none of the points on any of the lines. What is the homology of this space? [Hint: Mayer-Vietoris doesn't just compute the homology of $U \cup V$ from that of U, V and $U \cap V$, but it can also be used to compute the homology of $U \cap V$ from that of U, V and $U \cup V$.]

Let U be the complement of the n_0 points, let V be the complement of the n_1 lines, so $U \cup V = \mathbb{R}^3$ and $U \cap V$ is the space we're looking for. Since $\tilde{H}_k(U \cup V)$ is trivial, Mayer Vietoris (for reduced homology) says that $\tilde{H}_k(U \cap V) = \tilde{H}_k(U) \oplus \tilde{H}_k(V)$, which is $\mathbf{Z}^{n_1}$ if $k = 1$, $\mathbf{Z}^{n_0}$ if $k = 2$, and zero in all other dimensions. This implies that $H_k(U \cap V)$ is $\mathbf{Z}$ if $k = 0$, $\mathbf{Z}^{n_1}$ if $k = 1$, $\mathbf{Z}^{n_0}$ if $k = 2$ and zero if $k \geq 3$.