Algebraic Topology
Homework 12: Due Wednesday, November 24

Let $X_1, X_2, \ldots$ be a sequence of path-connected topological spaces, and suppose that $x_i \in X_i$ for each i. The *wedge* of the X's, denoted $\vee_i X_i$, is the quotient space of the disjoint union $\coprod_i X_i$ by the identification $x_i \sim x_j$ for all i, j. For instance, the wedge of two circles is a figure 8. The topology of $\vee_i X_i$ is that a set is open if (and only if) its intersection with each X_i is open. For the wedge of a finite number of spaces, this definition is obvious, but it is more subtle for infinite wedges. (The Hawaiian earring is *not* an infinite wedge of circles!)

Problem 1 Prove the following: If $Y = \vee_i X_i$ for a collection of spaces X_i, and if each x_i has a neighborhood that (strongly) deformation retracts to x_i, then $\tilde{H}_k(Y) = \oplus_i \tilde{H}_k(X_i)$. [Warning: as with Van Kampen's theorem, which says that the fundamental group of Y is the free product of the fundamental groups of the X_i's, the condition of having an appropriate neighborhood is crucial. For finite collections, this problem could be solved by induction and Mayer-Vietoris, but there is a better way. For infinite collections, don't even think about induction.]

Let U_i be a contractible set around x_i, and let $U = \coprod U_i$. Then $\coprod x_i$ is a deformation retract of $\coprod U_i$, and $(\coprod X_i, \coprod x_i)$ is a good pair, so $H_k(\coprod X_i, \coprod x_i) = \tilde{H}_k(Y)$. We then have an exact sequence $\tilde{H}_k(\coprod x_i) \to \tilde{H}_k(\coprod X_i) \to \tilde{H}_k(Y) \to \tilde{H}_{k-1}(\coprod x_i)$, etc. For $k \geq 1$, $\tilde{H}_k(\coprod x_i)$ is trivial, and the map $\tilde{H}_0(\coprod x_i) \to \tilde{H}_0(\coprod X_i)$ is an isomorphism. This implies that $\tilde{H}_0(Y) = 0 = \oplus \tilde{H}_0(X_i)$ and that, for $k > 0$, $\tilde{H}_k(Y) = \tilde{H}_k(\coprod X_i) = H_k(\coprod X_i) = \oplus H_k(X_i) = \oplus \tilde{H}_k(X_i)$.

Problem 2. Suppose that the topological spaces X and Y are path-connected. Show that $H_1(X \times Y)$ is isomorphic to $H_1(X) \oplus H_1(Y)$. [More generally, there is a simple formula for the free part of $H_n(X \times Y)$ in terms of the homology of X and Y, but the torsion part is more complicated. If H_k^f denotes the free part of H_k, then $H_n^f(X \times Y) = \oplus_k (H_k^f(X) \otimes H_{n-k}^f(Y))$. No, you are *not* expected to prove this!]

Since $\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y)$, the abelianization of $\pi_1(X \times Y)$ is the direct sum of the abelianization of $\pi_1(X)$ and the abelianization of $\pi_1(Y)$. But X, Y and $X \times Y$ are all path-connected, so the abelianization of π_1 in each case is H_1.

Since $H_n(S^n)$ is infinite cyclic, there are two choices of generators for $H_n(S^n)$. Each is called an *orientiation* of S^n. Given a choice of orientation, we can think of the top homology as the integers. If $f : S^n \to S^n$ is a continuous map between oriented spheres, then $f_* : \mathbf{Z} = H_n(S^n) \to \mathbf{Z} = H_n(S^n)$ is multiplication by an integer, which we call the *degree* of f. Note that, for maps from an n-sphere to itself, the degree doesn't depend on the orientation (since switching the orientation would involve multiplying by -1 twice), but for maps between distinct spheres, you have to pick orientations before you can define a degree. Note that $deg(f \circ g) = deg(f)deg(g)$ if both f and g are maps $S^n \to S^n$, and that $deg(f_0) = deg(f_1)$ if f_0 and f_1 are homotopic.

Problem 3. Consider the reflection map $R(x_1, x_2) = (-x_1, x_2)$ that maps $\mathbb{R}^2$ to itself, and also maps S^1 to itself. Show that R has degree -1. [Hint: Pick an explicit generator α of $H_1(S^1)$ such that $R_\#(\alpha) = -\alpha$.] Now consider reflection on the n-sphere, where $R(x_1, x_2, \ldots, x_{n+1}) = (-x_1, x_2, \ldots, x_{n+1})$. Show that R has degree -1.

For the circle, we saw that a generator of H_1 is a sum of two 1-cubes (a.k.a. paths), each running from $(-1,0)$ to $(1,0)$ along different semicircles. Rotate this picture by 90 degrees to get a generator that is the sum of two paths from $(0,-1)$ to $(0,1)$. Since R sends each path to the other, R_* is multiplication by -1 on $H_1(S^1)$.

The same idea works in higher dimensions. For S^n, think of the equator as the boundary of the n-cube, and a generator of S^n is the difference of two n-cubes with the same boundary, one running across the northern hemisphere and one running across the southern, with reflection interchanging these two cubes.

The same thing works for flipping the sign of x_k instead of flipping the sign of x_1. If you let R_k be the map that flips the k-th coordinate, and if S_{ij} just swaps the ith and jth coordinate, then $R_k = S_{jk} R_j S_{jk}^{-1}$, so R_k and R_j have the same degree, namely -1. Since the antipodal map is the product of $n+1$ reflections, it has degree $(-1)^{n+1}$.

Problem 4. Show that the antipodal map of S^n is homotopic to the identity if, and only if, n is odd.

If n is even, then the antipodal map has degree -1, and so can't be homotopic to the identity map, which has degree $+1$. If n is odd, then the map $f_t(x_1, \ldots, x_n) = (x_1 \cos(\pi t) - x_2 \sin(\pi t), x_2 \cos(\pi t) + x_1 \sin(\pi t), x_3 \cos(\pi t) - x_4 \sin(\pi t), \ldots)$ gives a homotopy between the identity map f_0 and the antipodal map f_1.

The van Kampen computation of π_1 of a surface, when Abelianized, essentially gives the Mayer-Vietoris computation of H_1 of the surface. The following problem generalizes that idea.

Problem 5. Let U, V, and $X = U \cup V$ be as in the setup for Van Kampen's theorem. That is, each is open and path-connected and contains the base point x_0. Without using homology, write down (and prove) an "Abelianized van Kampen's theorem" that relates $\pi^A(X)$ to $\pi^A(U)$, $\pi^A(V)$, $\pi^A(U \cap V)$ and the way these groups map to one another, where π^A denotes the Abelianization of π_1. (You can take the standard van Kampen's theorem as a given – there's no need to reprove that!) Then write down the (reduced) Mayer-Vietoris sequence that computes $H_1(X)$ from $H_1(U)$, $H_1(V)$ and $H_1(U \cap V)$, and show how the maps in the M-V sequence relate to the maps in your Abelianized van Kampen's theorem.

Theorem: There is an exact sequence

$$\pi'(U \cap V) \to \pi'(U) \oplus \pi'(V) \to \pi'(X) \to 0,$$

where the first map is $i_*(h) = (i_1)_*(h) - (i_2)_*(h)$, and i_1 and i_2 are the inclusions of $U \cap V$ into U and V, respectively.

Note that U, V, and $U \cap V$ are path connected, so they all have trivial $\tilde{H}_0$. van Kampen says that $\pi_1(X) = \pi_1(U) * \pi_1(V)/N$, where N is the normal subgroup generated by $(i_1)_*(h)((i_2)_*(h))^{-1}$, where h ranges over $\pi_1(U \cap V)$. Abelianizing everything, we get that $\pi'(X) = \pi'(U) \oplus \pi'(V)/N'$, where N' is the subgroup generated by $(i_1)_*(h) - (i_2)_*(h)$, where h ranges over $\pi'(U \cap V)$. That's exactly our theorem.

This greatly resembles the Mayer-Vietoris sequence in reduced homology:

$$H_1(U \cap V) \to H_1(U) \oplus H_1(V) \to H_1(X) \to \tilde{H}_0(U \cap V) = 0,$$

but the maps are slightly different. In the Abelianized van Kampen's theorem, the map from $\pi'(U \cap V) \to \pi'(U) \oplus \pi'(V)$ *is* $i_*(\alpha) = (i_1)_*(\alpha) - (i_2)_*(\alpha)$*, or* $i_*(\alpha) = ((i_1)_*(\alpha), -(i_2)_*(\alpha))$ *(if you view the direct sum as an ordered pair), while in Mayer Vietoris it is* $i_*(\alpha) = ((i_1)_*(\alpha), (i_2)_*(\alpha))$*. This sign difference is irrelevant, and is corrected by the sign difference in the map from the direct sum to* $\pi'(X)$ *or* $H_1(X)$*. In van Kampen, that map is a sum, while in Mayer-Vietoris it is a difference.*

Of course, the other distinctions are that Mayer-Vietoris says things about higher homology groups, not just H_1*, and that Mayer-Vietoris applies even when* U*,* V *and* $U \cap V$ *are not path-connected. [Note: If* $U \cap V$ *is not path connected, then* $\tilde{H}_0(U \cap V)$ *is nonzero and* $j_* : H_1(U) \oplus H_1(V) \to H_1(X)$ *isn't necessarily onto.]*

Book problems: Page 191, problem 2.5 and page 201, problem 3.7.

2.5: If f *is not onto, then there is a point* $p \in S^n$ *such that* f *maps to* $S^n - \{p\}$*. But* $S^n - \{p\}$ *is contractible, so* f *is homotopic to a constant map, hence has degree zero. Another way to see this is to say that we have maps* $\tilde{f} : S^n \to S^n - \{p\}$ *and* $i : S^n \to S^n - \{p\}$ *with* $f = i \circ \tilde{f}$*. Since* $H^n(S^n - \{p\})$ *vanishes,* $\tilde{f}_* = 0$*, so* $f_* = i_* \circ \tilde{f}_*$ *is also zero.*

3.7. The local homology at (x, y, z) *depends on how many variables are zero. If only one, then we have the local homology of a point in the plane, which is the reduced homology of a disk with boundary points identified, which is* $\mathbf{Z}$ *in dimension 2 and zero otherwise. If* $x = y = 0$ *and* $z \neq 0$*, then the local homology is the reduced homology of a space obtained by taking two intersecting disks and identifying all boundary points. This is the wedge of three spheres, and has* $H_2 = \mathbf{Z}^3$*. If* $x = y = z = 0$*, then we get the reduced homology of the intersection of three disks with all boundary points identified. This is homotopy equivalent to the wedge of seven spheres, and has* $H_2 = \mathbf{Z}^7$*. (The three planes divide the unit sphere into eight regions. Our homotopy equivalence involves opening one of these regions out and closing off the other seven. In general, if you have* $k \leq n$ *hyperplanes intersecting in the obvious way in* $\mathbb{R}^n$*, you'll get local homology* $H_{n-1} = \mathbf{Z}^{2^k - 1}$*. Hence* $\mathbf{Z}$*,* $\mathbf{Z}^3$ *and* $\mathbf{Z}^7$ *in* $\mathbb{R}^3$*.) Since the origin is the unique point with second local homology equal to* $\mathbf{Z}^7$*, any self-homeomorphism of* X *must take the origin to itself.*