Algebraic Topology

Homework 13: Due Friday, December 3

Problem 1. We saw in class how RP^2 is a CW complex with one 2-cell, one 1-cell, and one 0-cell, with $\partial e^{(2)} = 2e^{(1)}$ and $\partial e^{(1)} = 0$, which implies that $H_2(RP^2) = 0$, $H_1 = \mathbf{Z}_2$ and $H_0 = \mathbf{Z}$. Find a CW decomposition for $RP^2 \times RP^2$ and use it to compute the homology of $RP^2 \times RP^2$. This example provides a counterexample to an obvious guess of how the homology of $X \times Y$ is related to the homology of X and Y. The obvious formula works for the free part of the homology, but the torsion part behaves differently.

For no apparent reason, I'm going to switch from superscripts to subscripts. In RP^2, we have three cells, $e_{0,1,2}$, where e_i is i-dimensional. Let $e_{ij} = e_i \times e_j$ be an $i+j$-cell in $RP^2 \times RP^2$. Our complex has one 4-cell (e_{22}), two 3-cells (e_{21} and e_{12}), 3 2-cells (e_{20}, e_{11}, e_{02}), 2 1-cells (e_{10}, e_{01}) and one 0-cell e_{00}. The boundary of the product of two cubes $C_1 \times C_2$ is $(\partial C_1) \times C_2 + (-1)^d C_1 \times \partial C_2$, where d is the dimension of C_1. Since $\partial e_2 = 2e_1$ and $\partial e_1 = 0$, we have that $\partial e_{22} = 2e_{12} + 2e_{21}$, $\partial e_{21} = 2e_{11}$, $\partial e_{12} = -2e_{11}$, $\partial e_{20} = 2e_{10}$, $\partial e_{11} = 0$, $\partial e_{02} = 2e_{01}$, $\partial e_{10} = \partial e_{01} = \partial e_{00} = 0$. In terms of matrices, $\partial_4 = (2, 2)$, $\partial_3 = \begin{pmatrix} 0 & 0 \\ 2 & -2 \\ 0 & 0 \end{pmatrix}$, $\partial_2 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, $\partial_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. The kernel of ∂_4 is empty, so $H_4 = 0$. The kernel of ∂_3 is the span of $e_{21} + e_{12}$, while the image of ∂_4 is the span of $2e_{21} + 2e_{12}$, so $H_3 = \mathbf{Z}_2$. The kernel of ∂_2 is the span of e_{11}, while the image of ∂_3 is the span of $2e_{11}$, so $H_2 = \mathbf{Z}_2$. Likewise, $H_1 = \mathbf{Z}_2 \oplus \mathbf{Z}_2$ and $H_0 = \mathbf{Z}$.

Problem 2. Let X be the torus of revolution $(\sqrt{x^2 + z^2} - 2)^2 + y^2 = 1$. Let Y be the union of X with the two disks $(x-2)^2 + y^2 \leq 1, z = 0$ and $(x+2)^2 + y^2 \leq 1, z = 0$. Exhibit a CW structure for this torus-with-two-disks space (which appeared on last year's midterm), and use this structure to compute the homology of this space.

The minimal structure that I can think of involves two vertices v_{12}, paths e_i from v_i to itself, each looping around a meridian of the torus, two paths e_3 and e_4 from v_1 to v_2 (each being half of a longitude), and four faces, with f_1 and f_2 being disks bounded by e_1 and e_2, and with f_3 and f_4 each covering half of the original torus, and with $\partial f_3 = -e_1 + e_3 + e_2 - e_3 = -e_1 + e_2$, and with $\partial f_4 = e_1 - e_4 - e_2 + e_4 = e_1 - e_2$. Expressed as matrices, we have $\partial_2 = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and $\partial_1 = \begin{pmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$. The kernel of ∂_2 is 2-dimensional, and is spanned by $-f_1 + f_2 + f_4$ and $f_1 - f_2 + f_3$ (so $H_2 = \mathbf{Z}^2$), while the image is the span of e_1 and e_2. The kernel of ∂_1 is the span of e_1, e_2 and $e_3 - e_4$, so $H_1 = \mathbf{Z}$, and is generated by $e_3 - e_4$. $H_0 = \mathbf{Z}$, of course.

Problem 3. Construct a 3-dimensional CW complex (that is, at least one 3-cell and no cells of dimension greater than 3) with the following homology groups: $H_0 = \mathbf{Z}$, $H_1 = \mathbf{Z}_2 \oplus \mathbf{Z}_3$, $H_2 = \mathbf{Z}$, $H_3 = 0$. There are many, many possible answers.

My simplest example is the wedge of four spaces: a sphere, a closed 3-ball, an RP^2, and a space obtained by taking a circle and wrapping the boundary of a 2-cell 3 times around that circle. In other words, we have one vertex v, 2 edges $e_{1,2}$, 4 faces $f_{1,2,3,4}$, and one 3-cell s, with $\partial e_i = 0$, $\partial f_1 = 0$ $\partial f_2 = 2e_1$, $\partial f_3 = 3e_2$, $\partial f_4 = 0$, and $\partial s = f_4$. v and f_1 make the sphere, v, e_1, f_2 make the RP^2, v, e_2, f_3 make the space with $H_1 = \mathbf{Z}_3$, and v, f_4, s make a solid ball.

Problem 4. Let X be a simplicial complex with a finite number of simplices (aka a "finite simplicial complex"). Prove that the simplicial homology of X is isomorphic to the singular homology of X. [Hint: writing X as a CW complex is easy, and this gives a correspondence between simplicial chains and cellular chains. What takes work is showing that the boundary maps in the simplicial complex are the same as the maps in the cellular computation of the singular homology.]

Since an n-sphere is homeomorphic to an n-cube, the trick is finding an appropriate map from the unit n-cube $[0,1]^n$ to an n-simplex $\Delta_{v_0,\ldots,v_n}$, viewed as the convex hull of $n+1$ points $v_0, \ldots, v_n$. One possibility is the following recursive definition: $f_1(x_1) = x_1 v_1 + (1-x_1) v_0$, $f_n(x_1, \ldots, x_n) = x_n v_n + (1-x_n) f_{n-1}(x_1, \ldots, x_{n-1})$. Note that if any of the variables x_k are set equal to 1, then the map is independent of all of the previous variables $x_1, \ldots, x_{k-1}$. This implies that the only nondegenerate faces are $B_1, A_1, A_2, \ldots, A_n$. But B_1 is precisely the map to the simplex $\Delta_{v_1,\ldots,v_n}$, while A_k is the map to $\Delta_{v_0,\ldots,v_{k-1},v_{k+1},\ldots,v_n}$. Since B_1 appears with a positive sign in ∂f_n, and A_k appears with sign $(-1)^k$, this exactly reproduces the boundary in singular homology.

Let G_1, $G_2, \ldots$ be Abelian groups, and suppose that we have maps $\rho_n : G_n \to G_{n+1}$. The *direct limit* of the groups G_n is the quotient of the product $G_1 \times G_2 \times \cdots$ by the identification $x \sim y$ if $x \in G_n$, $y \in G_{n+1}$, and $y = \rho_n(x)$. That is, G_1 is identified with a subgroup of G_2, G_2 is identified with a subgroup of G_3, etc. The groups G_n are called *approximants* to the direct limit.

The direct limit of spaces is similar. Let $X_1, \ldots,$ be spaces, and consider continuous maps $f_n : X_n \to X_{n+1}$. As a set, the direct limit of the X_n's is the union $\coprod X_n$, modulo the identifications $x_n \sim f_n(x_n)$ if $x_n \in X_n$. A set is considered closed if its intersection with (the image of) each X_n is closed (in X_n).

The most common example is when $X_1 \subset X_2 \subset \cdots$, so the maps f_n are just inclusions. The direct limit is then the union, albeit with a particular topology. For instance, X_n might be the n-skeleton of a CW complex, or X_n might be a finite sub-complex of an infinite (but finite-dimensional) CW complex.

CW complexes have the following useful property that you are free to use below: Every compact subspace is contained in a finite sub-complex, and in particular in a finite skeleton X_n. A proof may be found on p520 of Hatcher, but here's a sketch. If a compact set C hits infinitely many cells, then we could find a subset $S = \{x_1, x_2, \ldots\}$ of C with each x_i in a different cell. Since S hits the interior of each cell at most once, it hits the closure of each cell only a finite number of times (I'm skipping some details here), so the intersection of S with each cell is closed, so S is closed. Being a closed subset of the compact C, S must be compact. But S has the discrete topology, and so is not compact. Contradiction.

Problem 5. Let X be the direct limit of a collection of spaces $X_1 \subset X_2 \subset \cdots$, such that any compact subset of X lies in some X_n. Prove that the homology of X is the direct limit of the homology of X_n.

There is a natural map from each X_n to X, which induces a map from the homology of X_n to the homology of X. Since the inclusion of X_n into X_m into X is the same as the inclusion of X_n into X, two classes that are equivalent in the direct limit get mapped to the same homology class in X. In other words, there is a map from the direct limit of $H_k(X_n)$ to $H_k(X)$. We must show that this map is 1-1 and onto.

First onto. Every homology class in X is represented by a finite chain. But the image of the cubes in that chain lies in a compact set, and so lives in a finite approximant X_n. Thus the homology class in X can be represented by a homology class in some X_n.

For 1-1, suppose that a homology class $[\alpha]$ in X_n maps to the trivial class in X. But that means that α is the boundary of some chain β in X. Since β lives in some other finite X_m (where m may be much greater than n), the image of $[\alpha]$ in $H_k(X_m)$ is trivial, and so $[\alpha]$ is trivial in the direct limit.

Problem 6. As a corollary to Problem 5, prove the following theorem from last week's lectures: Let X be a CW complex, and let F^n be a free abelian group whose generators are the n-cells of X. Then there exist maps $\partial_n : F^n \to F^{n-1}$ with $\partial^2 = 0$ such that the n-th singular homology of X is isomorphic to $ker(\partial_n)/Im(\partial_{n+1})$. (We can take $\partial_0 : F^0 \to 0$ to be the zero map.) When X is finite-dimensional, this was proved in class, and you are free to use those results without reproving them. The trick is extending this to infinite dimensions.

Thanks to the comment before Problem 5, we know that $H_k(X)$ is the direct limit of $H_k(X_n)$. However, for $n > k$, $H_k(X_n)$ is the kernel of ∂_k mod the image of ∂_{k+1}. (This is "theorem 2" in class). Furthermore, the inclusion $X_n \to X_{n+1}$ induces an isomorphism $H_k(X_{n+1}) = H_k(X_n)$ for all $k < n$. (This comes from the long exact sequence of the pair (X_{n+1}, X_n) and the fact that X_{n+1}/X_n is a wedge of $n+1$-spheres, so $H_k(X_{n+1}, X_n)$ is trivial for $k \leq n$.) The direct limit of the same group over and over again, via the identity map, is just the group itself.

Problem 7. The inclusion $\mathbb{R}^n \to \mathbb{R}^{n+1}$, where $(x_1, \ldots, x_n)$ goes to $(x_1, \ldots, x_n, 0)$, induces an inclusion of S^{n-1} into S^n. Let S^∞ be the direct limit of $S^1 \subset S^2 \subset \cdots$, and let RP^∞ be the quotient of S^∞ by the antipodal map. Let CP^∞ be the direct limit of the spaces CP^n induced by the obvious inclusion of each C^n in C^{n+1}. Compute the homologies of S^∞, RP^∞, and CP^∞.

$H_0(S^\infty) = \mathbf{Z}$, since S^∞ is path-connected. For $k > 0$, since $H_k(S^n) = 0$ for $n > k$, we have $H_k(S^\infty) = \lim H_k(S^n) = \lim 0 = 0$. Similarly, $H_k(RP^n)$ is either 0 or $\mathbf{Z}_2$ for $n > k$, depending on whether k is even or odd, so $H_k(RP^\infty)$ is $\mathbf{Z}_2$ when k is odd, $\mathbf{Z}$ when $k = 0$, and is trivial for even $k > 0$. Finally, $H_k(CP^n)$ is $\mathbf{Z}$ if $k \leq 2n$ is even and 0 otherwise. Taking a limit gives $H_k(CP^n) = \mathbf{Z}$ if $k \geq 0$ is even and is zero otherwise.

An aside about cohomology: By the universal coefficients theorem, $H^k(CP^n)$ is also $\mathbf{Z}$ when k is even and is trivial otherwise. If we let x be the generator of H^2, then x^j is the generator of H^{2j}. Thus, the cohomology *ring* of CP^∞ is just $\mathbf{Z}[x]$. The cohomology ring of CP^n is $\mathbf{Z}[x]/I$, where I is the ideal generated by x^{n+1}.