Algebraic Topology
Homework 2: Due September 9

Problem 1. Consider a hexagon with opposite edges identified, so that the pattern of edges around the boundary is $abca^{-1}b^{-1}c^{-1}$. Which vertices are identified? Using orientability and Euler characteristic, identify this space as either S^2, $\#_n T^2$, or $\#_n RP^2$. Then find an explicit set of cut-and-paste moves that turns the hexagon into the standard model for your answer.

Every other vertex is identified. Edge a runs from P to Q, b from Q to P, c from P to Q, a^{-1} from Q to P, b^{-1} from P to Q, and c^{-1} from Q to P. Since all edges are type 1, and since the Euler characteristic is 2-3+1=0, we must have the ordinary torus T^2.

As for the cut-and-paste moves, I apologize for not knowing how to draw figures in TeX in finite time. But here are the instructions, following the proof of Theorem 5.1. Let P_1 be the tail of the first a. Let d be the line from P_1 to the head of the first b, and let e be the line from P_1 to the head of a^{-1}. Cut along d and glue triangle abd^{-1} back along the other a to get the polygon $dcbd^{-1}b^{-1}c^{-1}$. This reduces the number of Q vertices from 3 to 2. Then cut along e and glue triangle $eb^{-1}c^{-1}$ along the other c to get $deb^{-1}bd^{-1}e^{-1}$, reducing the number of Q vertices to 1. Then glue b^{-1} and b together to get $ded^{-1}e^{-1}$, which is the canonical form for a torus.

Do problems 7.2 and 7.3 and 7.4 on page 19. Then follow the proof of Theorem 5.1 to convert each of these triangulations into a polygon of standard form. In each case, what surface are we looking at?

For these problems I will denote edges by the vertices that they run between, so (12) is the edge from 1 to 2, and $(21) = (12)^{-1}$.

7.2 is the tetrahedron that we did in class. Start with triangle 123, glue in 234 along (23). Glue in 341 along (34). Glue in 412 along (24). The result is a polygon with vertices 1,2,1,4,1,3 and edges (12), (21), (14), (41), (13), and (31). Apply the move of figure 1.17 twice to get a lune with vertices 1 and 2 and edges (12) and (21). This identifies to a sphere.

For 7.3, start with triangle 123. Glue in 234 along (23). Glue in 345 along (34). Glue in 451 along (45). Glue in 512 along (51). Glue in 136 along (13). Glue in 246 along (24). (We could just as well have glued it in along (46), but I chose to glue it along (24).) Glue in 356 along (35). Glue in 416 along (41). Glue in 526 along (52). The result is a polygon with vertices 1,2,6,4,6,1,2,6,5,6,3,6. The edges (64) and (46) cancel, as do (65) and (56), as do (63) and (36). This yields the polygon 126126, or in terms of edges, $abcabc$, where $a = (12)$, $b = (26)$ and $c = (61)$.

Now cut from the end of the first a to the end of the second, and glue the a's together to get $ddc^{-1}b^{-1}bc$. Sew up the $b^{-1}b$, and then sew up the resulting $c^{-1}c$, to get dd, which is the standard form for RP^2.

Unfortunately, that quick solution isn't following the steps of the proof of Theorem 5.1! Following the recipe in the proof would mean first cutting from the beginning of the first a to the end of the first b and gluing back along the other a to get $dcdb^{-1}bc$, then canceling the $b^{-1}b$ to get $dcdc$. This is the standard "square" picture of RP^2. To put it into canonical form, cut along a diagonal (say, from the beginning of a d to the beginning of the other d and glue back along c to get $eec^{-1}c$, hence ee, which is the canonical form for RP^2.

7.4 is trickier, since we can't place the triangles in the order given. 124 and 235 don't share an edge! Still, you can hunt around and always find a triangle to add until you're done. Here's what I did:

Start with 124, glue in 134 along (14), glue in 346 along (34), glue in 137 along (13), glue in 245 along (24), glue in 253 along (25), glue in 457 along (45), glue in 356 along (35), glue in 561 along (56), glue in 467 along (46), glue in 571 along (51), glue in 672 along (67), glue in 126 along (62), and finally glue in 237 along (73).

The result is a polygon with vertices 7123617574721632. In terms of edges, we have $abcdea^{-1}ff^{-1}gg^{-1}hb^{-1}e^{-1}d^{-1}c^{-1}h^{-1}$. Ugh! But it's not as bad as it looks.

We immediate cancel the ff^{-1} and gg^{-1} terms to get $abcdea^{-1}hb^{-1}e^{-1}d^{-1}c^{-1}h^{-1}$. Cut from the beginning of d to the end of e, and glue back in along d^{-1}, to get $abcia^{-1}hb^{-1}e^{-1}ei^{-1}c^{-1}h^{-1}$, and hence $abcia^{-1}hb^{-1}i^{-1}c^{-1}h^{-1}$. This "cut out a triangle, glue it back in and cancel an element with its inverse" is a general trick for replacing two consecutive edges $\alpha\beta$ with γ if they later appear as $\beta^{-1}\alpha^{-1}$, which then turns into γ^{-1}. (It's also the third step in the proof of 5.1, as we're reducing a class of vertices from 2 elements to 1, and then using the second step to eliminate the one.) Applying this trick again, we can replace $h^{-1}a$ with j to get $jbcij^{-1}b^{-1}i^{-1}c^{-1}$ and then replace ci with k to get $jbkj^{-1}b^{-1}k^{-1}$. But that's the situation of Problem 1, so we know how to reduce that to a standard torus.

8.1: Suppose we have m n-gons meeting at each vertex. We have $2e = nf = mv$, where v, e, and f count vertices, edges and faces. The Euler characteristic is then $e[(2/n)+(2/m)-1]$. Since this equals 2, we must have $(2/m)+(2/n) > 1$. At least one of m and n have to be less than 4, since $2/4+2/4$ is too small. If one is three, then the other is at most 5, since $2/3+2/6 = 1$. This means that the only possibilities are $n = m = 3$ (tetrahedron) or $n = 3$ and $m = 4$ (octahedron) or $n = 3$ and $m = 5$ (icosahedron) or $n = 4$ and $m = 3$ (cube) or $n = 5$ and $m = 3$ (dodecahedron).

8.8. All the vertices are identified, since the end of a_n is the beginning of both a_1 and a_1^{-1}, which then equals the beginning of a_2 and the beginning of a_2^{-1}, which then equals the beginning of a_3 and the beginning of a_3^{-1}, etc. Since there are n edges, one vertex, and a type-2 edge, this is a non-orientable surface of Euler characteristic $1+1-n = 2-n$, hence equals $\#_n RP^2$.

8.9. Note that all edges are type 1, so this surface is orientable. Each vertex is identified with a vertex $n+1$ steps around the $2n$-gon (e.g., beginning of a_1 equals end of a_1^{-1}). If n is even, this means that all the vertices are identified. If n is odd, this means that there are two classes of vertices. So the Euler characteristic for n even is $1-n+1 = 2-n$, and we have the connected sum of $n/2$ tori. The Euler characteristic for n odd is $2-n+1 = 3-n$, so we have the connected sum of $(n-1)/2$ tori. Problem 1 was an example of this with $n = 3$.