Algebraic Topology
Homework 3: Due September 15

1. Page 42, problem 3.2

The path from x to y doesn't matter if π_1 is Abelian. The path does matter if π_1 is non-Abelian. To see this, note that the difference between the map induced by two paths γ, γ' from x to y is conjugation by $\gamma^{-1}\gamma'$. Specifically, if $f, g : \pi_1(X, x) \to \pi_1(X, y)$ are the two maps, then for each $\alpha \in \pi_1(X, x)$, $f(\alpha) = \gamma^{-1}\gamma' g(\alpha)\gamma'^{-1}\gamma$. If π_1 is abelian, this is the same as $g(\alpha)$. If π_1 isn't abelian, we can take $y = x$, γ to be the identity, and γ' to be any element of π_1 that isn't in the center of the group.

2. Page 46, problem 4.9. You may want to do 4.8 as a warm-up, and should assume that S is *not* a sphere.

Represent S as a polygon with sides identified in the canonical way, and suppose that the origin lies in the interior of the polygon. Let the origin be the point that's removed. Then take A to be the boundary of the polygon (with the usual identifications). The retraction map is just moving radially outwards until you hit the boundary. Since the polygon is in canonical form, with all vertices identified, its boundary is a union of finitely many circles, all meeting at a single point.

Note that this argument fails if S is a sphere, since in that case the boundary lune has two distinct vertices. Not only is the argument wrong, but the conclusion is wrong as well. The complement of a point in the 2-sphere is contractible, and hence cannot retract to a set that has a non-trivial fundamental group, and in particular cannot retract to a finite union of circles.

3. Page 49, problem 5.1

First the restatements: (1) If $X = U \cup V$, with U and V open and simply-connected, and with $U \cap V$ path-connected, then X is simply-connected. (2) If $X = \cup_i U_i$, with $U_1 \subset U_2 \subset \cdots$ and with each U_i nonempty, open, and simply connected, then X is simply connected.

Proof. Let α be a loop that represents an element of $\pi_1(X, x_0)$. The sets $V_i = \alpha^{-1}(U_i)$ form an open cover of $[0, 1]$, and this open cover has a Lebesgue number, i.e. a number ϵ so that every interval of length ϵ or less is contained in a single V_i. We can therefore break the interval $[0, 1]$ into a finite number of sub-intervals $[t_i, t_{i+1}]$ such that each such sub-interval maps to a single U_j. Let $t_0 = 0$ and $t_n = 1$.

Let α_i be the path obtained by restricting α to the interval $[t_{i-1}, t_i]$. If α_i lies in U_j and α_{i+1} lies in U_k (where U_k may or may not be different from U_j), then $\alpha(t_i)$ lies in $U_j \cap U_k$, which is path-connected, by assumption. Let β_i be a path from $\alpha(t_i)$ to x_0 that lies completely in $U_j \cap U_k$. We can take β_0 and β_n to be trivial paths.

The path α is homotopic to the path $\beta_0^{-1}\alpha_1\beta_1\beta_1^{-1}\alpha_2 \cdots \beta_{n-1}^{-1}\alpha_n\beta_n$. But this is a product of finitely many loops, $\beta_{i-1}^{-1}\alpha_i\beta_i$, each of which lies entirely in a single U_k. Since each U_k is simply connected, each of these loops can be homotoped to a constant. Since there are only finitely many such loops, we can homotope the entire path to a constant map.

4. Page 50, problem 5.2

Take U to be the complement of the north pole and V to be the complement of the south pole. $U \cap V$ is path-connected (it's an interval times the equator), and U and V are each homeomorphic to $\mathbb{R}^n$ and so are contractible, hence simply connected.

5. Page 50, problem 5.3.

The unit $n-1$-sphere is a deformation retract of $\mathbb{R}^n$ with the origin removed. By problem 5, this means that $\mathbb{R}^n$ minus the origin is simply connected if $n > 2$. But $\mathbb{R}^2$ minus a point has the same fundamental group as a circle, which is not trivial. Thus $\mathbb{R}^2$ minus a point is not homeomorphic to $\mathbb{R}^n$ minus a point, so so $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$.

Problem 6. Let X be any topological space. (If you want, you can restrict your attention to Hausdorff spaces, but it really doesn't matter.) let $Y = [0,1] \times X/\sim$, where $(0,x) \sim (0,y)$ for all $x, y \in X$. Y is called the *cone* of X, and the equivalence class of $(0,x)$ is called the *cone point*. Prove that Y is path-connected (easy) and simply connected.

Path connected: If $p = (s,x)$, then take the path $\alpha(t) = (st, x)$ from the cone point to p. Applying this path to all points simultaneously gives a homotopy between the identity and the map that sends all of Y to the cone point. In other words, the cone is contractible, hence simply connected.

Problem 7. Let X be any topological space, and let $Y = [0,1] \times X/\sim$, where $(0,x) \sim (0,y)$ and $(1,x) \sim (1,y)$ for all $x, y \in X$. Y is called the *(free) suspension* of X, and is sometimes denoted SX. (The *reduced suspension* ΣX is a slightly different space that doesn't concern us here. Look up Suspension (topology) on Wikipedia if you want the definition.) For instance, S^{k+1} is the suspension of S^k. Prove that if X is path-connected, then Y is simply-connected.

This is just a combination of problems 5.1 and 7. The complement of $(0,x)$ deformation retracts onto $(1,x)$, and so is simply connected. The complement of $(1,x)$ deformation retracts onto $(0,x)$, and is likewise simply connected. Their intersection deformation retracts to $\{1/2\} \times X$, and is path-connected.