Algebraic Topology
Homework 4 Solutions

1. *Page 53, problem 7.3. Note that "infinite product" means "with the product topology". Except where specifically noted, infinite products always have the product topology.*

Let $X = \prod X_i$, and let $\pi_i : X \to X_i$ be the projection on the i-th factor. Note that a map $\gamma : [0,1] \to \prod X_i$ is continuous if and only if each map $\gamma_i = \pi_i \circ \gamma : [0,1] \to X_i$ is continuous. If $\gamma \sim \gamma'$ via the homotopy Γ, then $\gamma_i \sim \gamma_i'$ via the homotopy $\pi_i \circ \Gamma$. If each $\gamma_i \sim \gamma_i'$ with homotopies Γ_i, then $\gamma \sim \gamma'$ with homotopy $\Gamma(s,t) = (\Gamma_1(s,t), \Gamma_2(s,t), \ldots)$.

In other words, a homotopy class of loops in X is a seqence of homotopy classes of loops in X_i, or $\pi_1(X,x) = \prod \pi_1(X_i, x_i)$, where $x = (x_1, x_2, \ldots)$. (I've used notation as if X is a countable product of topological spaces, but in fact the same arguments work for arbitrary products.)

2. *To see how evil alternatives to the product topology are, consider $X = S^1 \times S^1 \times \cdots$ with the box topology, where for definiteness we're taking S^1 to be the unit circle in the complex plane. (In the box topology, the set $U_1 \times U_2 \times \cdots$ is open if each U_i is open. Those infinite boxes form a base for the topology.) Give a characterization of all the continuous loops in X, and use this characterization to compute $\pi_1(X,x)$, where $x = (1,1,1,\ldots)$.*

A path γ is continuous if all components γ_i are continuous and all but finitely many of the γ_i's are *constant*. To see this, suppose that infinitely many γ_i's are non-constant, and WLOG we can assume that $\gamma_1, \gamma_2, \ldots$ are non-constant. There is a point $t_0 \in [0,1]$ such that infinitely many γ_i's are non-constant on every neighborhood of t, and WLOG we can number these $\gamma_1, \gamma_2, \ldots$. (Otherwise every point would have a neighborhood where only finitely may γ_i's vary. But a finite number of these neighborhoods cover the compact set $[0,1]$, so only finitely many γ_i's would vary in all.)

Pick a sequence of positive numbers δ_i that converge to zero. For each δ_i, one can find a point t_i with $|t_i - t_0| < \delta_i$ and $\gamma_i(t_i) \neq \gamma_i(t_0)$. Let ϵ_i be half the distance from $\gamma_i(t_i)$ to $\gamma_i(t_0)$, and let $U_i \subset S^1$ be the set of points whose distance to $\gamma_i(t_0)$ is less than ϵ_i. Note that $t_i \notin \gamma_i^{-1}(U_i)$, but that $t_0 \in \gamma_i^{-1}(U_i)$. If γ were continuous, then the preimage of $U = U_1 \times U_2 \times \cdots$ would be open, but this contradicts the fact that t_0 is in the preimage of U but points arbitrarily close to t_0 are not.

Since the only loops that we need to consider are those that affect only a finite number of factors, $\pi_1(X,x)$ is the *weak* product (aka direct sum) of an infinite number of copies of $\pi(S^1) = Z$.

3. *Let $Z[1/2]$ denote the set of rational numbers whose denominators are powers of 2. (This includes integers, since $1 = 2^0$.) Show that $Z[1/2]$ is an Abelian group under addition. Then show, in two ways, that $Z[1/2]$ is not finitely generated: First show this directly, by showing that the subgroup generated by any finite set of elements of $Z[1/2]$ cannot be the entire group. Second, show that $Z[1/2]$ does not satisfy the conclusions of Theorem 3.6 on page 70 (the classification theorem for finitely generated Abelian groups).*

We have the (commutative and associative) addition rule $a/2^n + b/2^m = (a2^m + b2^n)/2^{n+m}$, 0 is the additive identity, and $-a/2^n$ is the inverse of $a/2^n$. This shows that

$Z[1/2]$ is a group.

Now suppose that $a_1/2^{n_1}, \ldots, a_k/2^{n_k}$ are elements of $Z[1/2]$. These are all integers divided by 2^n, where n is the largest of the n_i's. The subgroup of $Z[1/2]$ that they generate is contained in $2^{-n}Z$, and in particular does not include $2^{-(n+1)}$. This shows that $Z[1/2]$ is not finitely generated.

For a different argument, suppose that $Z[1/2]$ is finitely generated. Since it has no torsion elements (being a subgroup of the rationals), Theorem 3.6 says that it must equal Z^n for some n. But Z^n has some elements that are not divisible by 2, while all elements of $Z[1/2]$ are divisible by 2.

4. *The classification theorem for finitely generated Abelian groups is stated in different ways in different books. One formulation says that each such group can be written as $Z^n \oplus Z_{a_1} \oplus \cdots \oplus Z_{a_k}$, where each a_i is a (positive) power of a prime, and that the numbers a_i are unique, up to permutation. Another formulation says that the group can be written as $Z^n \oplus Z_{t_1} \oplus \cdots \oplus Z_{t_\ell}$, where each t_i divides t_{i+1}, and that the numbers t_i are unique. Give an algorithm for converting from each of these canonical forms to the other. Then use this algorithm to show that each formulation of the theorem implies the other.*

Recall that $Z_m \times Z_n = Z_{nm}$ if (and only if) n and m are relatively prime. To convert from the t_i's to the a_i's, just write each t_i as a product of powers of primes, and hence Z_{t_i} as a product of cyclic groups whose orders are powers of primes. To convert from a_i's to t_i's, suppose that powers of p appear exactly k times, and that powers of all other primes appear at most k times. Let t_k be the product of the highest powers of each prime, t_{k-1} the product of the next highest powers, etc. For instance, $Z_9 \times Z_3 \times Z_8 \times Z_8 \times Z_7 = Z_{3\times 8} \times Z_{7\times 8\times 9}$. These procedures are clearly inverses of one another, so existence and uniqueness of one form implies existence and uniqueness of the other.

5. *Let C be a category. Two objects X, Y in C are called "isomorphic" if there are morphisms $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ is the identity morphism on Y and $g \circ f$ is the identity on X. An object Z in C is called "initial" or "universal" if, for every object Y in C, there is a unique morphism $Z \to Y$. For instance, in the category of groups and group homomorphisms, the trivial group is universal. In the category of sets and functions, the empty set is universal.*

Show that any two universal objects are isomorphic, and show that any object that's isomorphic to a universal object is also universal. In other words, the universal object is "unique up to isomorphism".

Suppose U and U' are universal. Since U is universal, there is an element $f \in Hom(U, U')$, and since U' is universal there is an element $g \in Hom(U', U)$. Now $f \circ g \in Hom(U', U')$. But since U' is universal, there is only one element of $Hom(U', U')$, so $f \circ g$ must be the identity. Ditto for $g \circ f$. U and U' are thus isomorphic.

If U'' is isomorphic to U, there are morphisms $\alpha \in Hom(U, U'')$ and $\beta \in Hom(U'', U)$ that are inverses. Let X be any object. Since there exists a morphism $\gamma : U \to X$, there exists a morphism $\gamma \circ \beta : U'' \to X$. If μ and ν are morphisms $U'' \to X$, then $\mu \circ \alpha$ and $\nu \circ \alpha$ are in $Hom(U, X)$, and thus have to be the same. But then $\mu = (\mu \circ \alpha) \circ \beta = (\nu \circ \alpha) \circ \beta = \nu$.

Thus U'' is universal.

[The dual of a universal object is a "terminal" or "final" object. An object T is final if, for each object Y, there is a unique morphism $Y \to T$. The proof you just gave, with arrows reversed, shows that final objects are unique up to isomorphism.

To every category C there is an opposite category C^{op}, such that the objects of C^{op} are the same as the objects of C, and such that $Hom_{C^{op}}(X, Y) = Hom_C(Y, X)$. In other words, the morphisms are the same as in C, only with all the arrows reversed. Compositions are defined in the opposite order. Universal objects in C become final objects in C^{op}, and vice-versa.]

6. *Find examples of categories with each of the following properties: (1) a universal object but no final object. (2) no universal object and no final object, (3) a universal object and a final object, with those objects not isomorphic, and (4) universal and final objects that are the same. (Objects that are both universal and final are called "zero objects")*

(1) Take a category whose objects are the real numbers $[0, 1)$, with a (unique) morphism $x \to y$ if and only if $x \leq y$. You could do the same thing for any partially ordered set with a minimal element. (2) Just do the same with a partially ordered set with no minimal element and no maximal element, like the integers, or the interval $(0, 1)$. (3) Use the closed interval $[0, 1]$, or just the 2-point set $\{0, 1\}$ with a morphism $0 \to 1$ but no morphism $1 \to 0$. (4) In the category of groups and group homomorphisms, the trivial (or zero) group is both universal (there is exactly one homomorphism from the trivial group to your favorite group) and final. This is where the term "zero object" comes from.

7. *Let X be a topological space. We build a category out of X whose objects are the open sets in X. If U and V are open sets, then $Hom(U, V)$ is empty unless $U \subset V$, in which case it consists of the inclusion map of U into V. In this category, what are the universal object(s) and final object(s)? (Justify your answers, of course.)*

The universal object is the empty set, and the final object is X itself, both viewed as open sets. The empty set is a subset of all open sets, and all open sets are subsets of X.