Algebraic Topology
Homework 5: Due October 1

1. Page 74, problems 4.1, 4.3, 4.4, 4.9.

4.1: Let $a \in G_1$ and $b \in G_2$ be nontrivial elements. ab and ba are both in reduced form, and are different words, so the free group is non-Abelian. Also, $(ab)^n = abab \dots ab \neq 1$, so ab has infinite order. Let $W = g_1 \dots g_n$ be an arbitrary non-empty word, with $g_1 \in G_i$ and $g_n \in G_j$. If $G_i \neq G_j$, then $g_1^{-1}W = g_2 \cdots g_n \neq W g_1^{-1} = g_1 \cdots g_n g_1^{-1}$, so W is not in the center. If $i = j$, then pick a nontrivial element $c \in G_k$ with $k \neq i$, and note that $cW \neq Wc$. Thus W is not in the center of the group.

4.3. Let $\psi_i = \phi_i \circ f_i$. Since G is the free product of the G_i's, there is a unique map $f : G \to G$ such that $f \circ \phi_i = \psi_i$, which is to say that there is a unique f that makes the diagram commute. The reduced word $W = g_1 g_2 \cdots g_n$ maps to the product $W' = f_{i_1}(g_1) f_{i_2}(g_2) \cdots f_{i_n}(g_n)$. If each f_i is injective, then each $f_i(g_i)$ is not the identity, so W' is a reduced word that is not the identity, so f is injective. If each f_i is surjective, then every possible reduced word $W' = g_1' \cdots g_n'$ is in the image of f: just take g_i to be in the preimage of g_i'.

4.4. As suggested, we prove this by induction on the length of the word. The base case (length 1) is a tautology. Suppose that $x = g_1 \cdots g_n$ has g_1 and g_n living in different groups. Then x manifestly has infinite order, as x^n is just x repeated n times. So, if x has finite order, then g_1 must live in the same group as g_n. But x is then conjugate to $g_2 \cdots (g_n g_1)$ which has length one less than that of x (or two less, if $g_n g_1$ is the identity). By the inductive hypothesis, x is then conjugate to an element of G or H.

2. Page 77, problem 5.3.

Let $g = g_1 \cdots g_n$ and let $h = h_1 \cdots h_m$. If g_1 and g_n live in the same group, then replace g with $g' = g_2 \cdots (g_n g_1)$ as a representative of its conjugacy class (or $g' = g_2 \cdots g_{n-1}$ if $g_n g_1$ is the identity.) Repeat until the first and last letters of g live in different groups, and do the same for h. After these procedures, the resulting words are conjugate if and only if they are cyclic permutations of one another.

The "if" part is clear, since you can induce a cyclic permutation by conjugating by a prefix. We must show that, if g_1 and g_n are in different groups, and if h_1 and h_m are in different groups, and if the g_i's aren't cyclic permutations of the h_i's, then g and h are not conjugate. Suppose that $m \geq n$ (the other case is similar), and that $g = WhW^{-1}$. We show that W cannot exist by induction on the length of W. We can assume that the last letter of W is neither h_1^{-1} nor h_m, since that letter would just induce a cyclic permutation and could be absorbed into the choice of h. But then the length of WhW^{-1} is longer than the length of h, which is a contradiction.

3. Page 81, problem 6.1

If $G_1 = \langle S_1 | R_1 \rangle$, where R_1 is a set of relations, and if $G_2 = \langle S_2, R_2 \rangle$, let $R_3 = \{[s_1, s_2] | s_1 \in S_1, s_2 \in S_2\}$, $R_4 = \{[s_1, s_1'] | s_1, s_1' \in S_1\}$. Modding out by R_3 says that the generators of G_1 commute with the generators of G_2, and hence that arbitrary elements of G_1 commute with arbitrary elements of G_2. Modding out by R_4 says that all elements of

G_1 commute. Then $G_1 * G_2 = \langle S_1, S_2 | R_1, R_2 \rangle$, $G_1/[G_1, G_1] = \langle S_1 | R_1, R_4 \rangle$, and $G_1 \times G_2 = \langle S_1, S_2 | R_1, R_2, R_3 \rangle$.

4. In class, I sketched an argument for the claim that the set of all reduced words is a group. Flesh out this argument by defining the product of two arbitrary reduced words and showing that this product is associative.

We can define the product recursively as follows: If g_n and h_1 belong to different groups, then the product of $g_1 \cdots g_n$ and $h_1 \cdots h_m$ is the concatenation $g_1 \cdots g_n h_1 \cdots h_m$. If g_n and h_1 belong to the same group and aren't inverses, then the product is $g_1 \cdots g_{n-1}(g_n h_1) h_2 \cdots h_m$. If g_n and h_1 belong to the same group and are inverses, then the product of $g_1 \cdots g_n$ and $h_1 \cdots h_n$ is the product of $g_1 \cdots g_{n-1}$ and $h_2 \cdots h_n$. This implies that if the last k terms of g are the inverses of the first k terms of h, and if g_{n-k} is not the inverse of h_{k+1}, then gh is equal to either $g_1 \cdots g_{n-k} h_{k+1} \cdots h_n$ or to $g_1 \cdots g_{n-1-k}(g_{n-k} h_{k+1}) h_{k+2} \cdots h_n$, depending on whether g_{n-k} and h_{k+1} belong to the same or different groups. I'll subsequently write them the first way, with the understanding that g_{n-k} and h_{k+1} need to be multiplied if they belong to the same group.

Now suppose that $g = g_1 \cdots g_n$, $h = h_1 \cdots h_m$ and $k = k_1 \cdots k_p$. We must show that $(gh)(k) = g(hk)$. Suppose that the first m_1 terms of h are inverses of the last m_1 terms of g and that $h_{m_1+1} \neq g_{n-m_1}^{-1}$, and suppose that the last m_2 terms of h are inverses of the first m_2 terms of h, and that $h_{m-m_2} \neq k_{m_2+1}^{-1}$.

If $m_1 + m_2 < m - 1$, $g(hk) = (g_1 \cdots g_n)(h_1 \cdots h_{m-m_2} k_{m_2+1} \cdots k_p)$, which equals $g_1 \cdots g_{n-m_1} h_{m_1+1} \cdots h_{m-m_2} k_{m_2+1} \cdots k_p$, which equals $(g_1 \cdots g_{n-m_1} h_{m_1+1} \cdots h_m)(k_1 \cdots k_p) = (gh)k$. The sub-word $h_{m_1+1} \cdots h_{m-m_2}$ serves to insulate the effects on h of left multiplication by g and right multiplication by k.

If $m_1 + m_2 > m$, then there are some letters in h that are inverses of both letters in g and letters in k, hence that $k_i = g_{n-m+i}$ for $m - m_1 < i \leq m_2$. That is, we can write $g = u_1 u_2 u_3$, $h = u_3^{-1} u_2^{-1} v$, $k = v^{-1} u_2 w$, where u_1 and v have no cancellations and where u_3^{-1} and w have no cancellations. Then $hk = u_3^{-1} w$ and $g(hk) = u_1 u_2 w$, while $gh = u_1 v$ and $(gh)k = u_1 u_2 w$.

The tricky cases are when $m_1 + m_2 = m$ or $m - 1$, as then we have to keep track of whether certain elements are in the same group or not.

If $m_1 + m_2 = m$, then I claim that both $g(hk)$ and $(gh)k$ are equal to the product of $g_1 \cdots g_{n-m_1}$ and $k_{m_2+1} \cdots k_p$. Not that it's equal to the concatenation, since g_{n-m_1} might be $k_{m_2+1}^{-1}$, just that it's equal to the product. To see this, note that $hk = h_1 \cdots h_{m_1} k_{m_2+1} \cdots k_p$, so that, by the recursive definition of product, $g(hk) = (g_1 \cdots g_{n-m_1} g_{n+1-m_1})(h_{m_1} k_{m_2+1} \cdots k_p) = (g_1 \cdots g_{n-m_1} h_{m_1}^{-1})(h_{m_1} k_{m_2+1} \cdots k_p)$. If h_{m_1} and k_{m_2+1} belong to different groups, then we can cancel $h_{m_1}^{-1}$ and h_{m_1}. If h_{m_1} and k_{m_2+1} belong to the same group, then we multiply $h_{m_1}^{-1}$ by $h_{m_1} k_{m_2+1}$ to get k_{m_2+1}. Either way, $g(hk)$ is of the claimed form. Likewise, $(gh)k = (g_1 \cdots g_{n-m_1} h_{m_1+1} \cdots h_m)(k_1 \cdots k_p) = (g_1 \cdots g_{n-m_1} h_{m_1+1})(k_{m_2} \cdots k_p)$. If g_{n-m_1} and h_{m_1+1} are of different groups, then h_{m_1+1} and k_{m_2} cancel. Otherwise, we compute $(g_{n-m_1} h_{m_1+1}) k_{m_2} = g_{n-m_1}$.

Finally, if $m_1 + m_2 = m - 1$, then $(gh)k$ and $g(hk)$ are both equal to the reduced word

$g_1 \cdots g_{n-m_1} h_{m_1+1} k_{m_2+1} \cdots k_p$ (where we may need to multiply some of g_{n-m_1}, h_{m_1+1} and k_{m_2+1} if they belong to the same groups) unless g_{n-m_1}, h_{m_1+1} and k_{m_2+1} all belong to the same group and their product is the identity, in which case we get the product of $g_1 \cdots g_{n-1-m_1}$ and $k_{m_2+1} \cdots k_p$ (which may or may not be the concatenation of the two).

5. Suppose we have a group G and several elements $\{g_i\}$ in G. Show that there exists a normal subgroup K of G containing all the g_i's, such that K is contained in every normal subgroup that contains the g_i's. This is called the normal subgroup generated by the g_i's. *Suppose that H is another group and $f : G \to H$ is a homomorphism. Let $p : G \to G/K$ be the obvious projection. Show that f lifts to a map $\hat{f} : G/K \to H$ if and only if every $f(g_i)$ is the identity. (By "lift to a map" I mean that $\hat{f}$ exists such that $f = \hat{f} \circ p$.)*

To see existence, let $K = \cap_j K_j$, where the K_j's are all the normal subgroups of G that contain all the g_i's. The arbitrary intersection of normal subgroups is normal, and every K_j contains K. If f maps all of the g_i's to the identity, then all of the g_i's are in the kernel of f. But the kernel is normal, so the kernel contains K, so f takes on the same value on every element of the coset aK. Let $\hat{f}(aK) = f(a)$. Conversely, if $\hat{f}$ exists, then $f(g_i) = \hat{f}(p(g_i)) = \hat{f}(e) = e$, where e denotes the identity in all groups.

6. Let G_1 and G_2 be two groups, and let H inject in both of them via injections i_1 and i_2. The amalgamated free product of G_1 and G_2 over H, denoted $G_1 *_H G_2$, is the quotient of $G_1 * G_2$ by the normal subgroup generated by $i_1(h)i_2(h)^{-1}$, where h ranges over H. It's like the free product of G_1 and G_2, only with $i_1(H)$ identified with $i_2(H)$. Let $\phi_1 : G_1 \to G_1 *_H G_2$ and $\phi_2 : G_2 \to G_1 *_H G_2$ be the obvious injections.

Suppose we have a group B and maps $\psi_1 : G_1 \to B$ and $\psi_2 : G_2 \to B$ such that $\psi_1 \circ i_1 = \psi_2 \circ i_2$. Show that there exists a unique homomorphism $f : G_1 *_H G_2 \to B$ such that $\psi_1 = f \circ \phi_1$ and $\psi_2 = f \circ \phi_2$.

Given maps ψ_1 and ψ_2, we have a unique map $f_0 : G_1 * G_2 \to H$ by the universal property of $G_1 * G_2$. Note that $i_1(h)i_2(h)^{-1}$ is in the kernel of f_0, since $\psi_1 \circ i_1(h) = \psi_2 \circ i_2(h)$. By problem 4, this means that f_0 lifts (uniquely) to a map f from $G_1 * G_2/K$ to B, where K is the normal subgroup generated by the $i_1(h)i_2(h)^{-1}$'s. In other words, we have a unique map $f : G_1 *_H G_2 \to B$.

7. Define a relevant category for which $G_1 *_H G_2$ is the universal object. In other words, express the conclusion of problem 5 as a universal property.

Let the objects of this category be groups A, together with homomorphisms $\alpha_1 : G_1 \to A$, $\alpha_2 : G_2 \to A$ such that $\alpha_1 \circ i_1 = \alpha_2 \circ i_2$. Let the morphisms be group homomorphisms $f : A \to B$ such that $\beta_1 = f \circ \alpha_1$, $\beta_2 = f \circ \alpha_2$. The universal object in this category is $G_1 *_H G_2$ together with the maps ϕ_1 and ϕ_2.

8. Repeat problems 6 and 7, only with i_1 and i_2 no longer assumed to be injective. For this problem, they're just group homomorphisms. Note that ϕ_1 and ϕ_2 are no longer necessarily injective. As far as I know, there is no standard term for the thing that replaces $G_1 *_H G_2$ – let's call it the generalized amalgamated product and denote it $G_1 \tilde{*}_H G_2$. (Some authors do use "amalgamated free product" to mean "generalized amalgamated free product", and they denote it $G_1 *_H G_2$, but others reserve the term for the case where H

is a subgroup of G_1 and also a subgroup of G_2.)

For problem 6, everything goes through exactly as before. We never used the fact that i_1 and i_2 were injective! For problem 7, we could do the same as before, but it's sometimes convenient to define the category in terms of a group A and a *triple* of maps $\alpha_1 : G_1 \to A$, $\alpha_2 : G_2 \to A$ and $\alpha_3 : H \to A$ with $\alpha_3 = \alpha_1 \circ i_1 = \alpha_2 \circ i_2$. That really isn't any different, since once you have α_1 and α_2 with $\alpha_1 \circ i_1 = \alpha_2 \circ i_2$, then α_3 is determined. See pages 87, 91 and 95 for diagrams that show up in this context, with the book using the letters ϕ and ψ and H where I used i and α and A.

The construction of Problem 8 is extremely important, thanks to van Kampen's theorem (aka the Seifert-van Kampen theorem), which says that the the fundamental group of the union of two open sets U and V is the generalized amalgamated free product $\pi_1(U, x_0)$ and $\pi_1(V, x_0)$ over $\pi_1(U \cap V, x_0)$. where $U \cap V$ is assumed path-connected, $x_0 \in U \cap V$, and the maps i_1 and i_2 are induced from the inclusions $U \cap V \to U$ and $U \cap V \to V$. We're going to spend a lot of time trying to understand the topology of this in Chapter 4. Chapter 3 is all about setting up the necessary algebra.

9. A free abelian group on 3 generators cannot inject in a free abelian group on two generators, but non-Abelian free groups are different. Let F_2 be the free group on two generators a, b, and let F_3 be the free group on generators s_1, s_2, s_3. Consider the map $f : F_3 \to F_2$ defined by $f(s_1) = ab$, $f(s_2) = a^2b^2$, $f(s_3) = a^3b^3$. Show that f is an injection.

I claim that, given a word w of length $n > 0$ in $s_1, s_2, s_3, s_1^{-1}, s_2^{-1}, s_3^{-1}$ (with s_i and s_i^{-1} not consecutive), that $f(w)$ is a word of length at least $n+1$ in powers of a and b, and is therefore not the empty word. This is clearly true for $n = 1$. So suppose that $w = w_0 s_i$ where w_0 is a word of length $n-1$ that does not end in s_i^{-1}. If w_0 ends in s_1, s_2 or s_3, then $f(w_0)$ ends in b and $f(w) = f(w_0)a^i b^i$ is longer than $f(w)$. If w_0 ends in s_j^{-1}, with $j \neq i$, then $f(w_0)$ ends in a^{-j}. But $a^{-j} f(s_i) = a^{i-j} b^i$ is longer than a^{-j}, so $f(w)$ is longer than $f(w_0)$. A similar argument involving powers of b (instead of powers of a) applies to words of the form $w = w_0 s_i^{-1}$.

10. Generalize the construction of problem 8 to construct a subgroup of F_2 that is not finitely generated. (You can take as given the fact that a free group on infinitely many generators is not finitely generated.)

Let $S = \{s_1, \ldots, \}$ be a countable set, and let $f : F_S \to F_2$ be the map defined by $f(s_i) = a^i b^i$. By the exact same argument as for problem 8, the image of a word of length n has length at least $n+1$, and so is not the empty word. Thus f is injective.