Algebraic Topology

Solutions to Homework 6: Due October 8

1. Page 94, problems 3.1 and 3.2. Both of these were essentially done in class, but spell them out anyway.

3.1: The only inclusions are $W \in V_i$. Since $\pi_1(W)$ is trivial, any map $\pi_1(V_i) \to H$ can be made into a commutative triangle via the trivial maps $\pi_1(W) \to \pi_1(V_i)$ and $\pi_1(W) \to H$. Theorem 2.2 therefore says that, given any collection of maps $\rho_i : \pi_1(V_i) \to H$, there exists a unique map $\sigma : \pi(X) \to H$ such that $\rho_i = \sigma \circ \psi_i$, where $\psi_i : \pi_1(V_i) \to \pi_1(X)$ is the map induced by inclusion. But this is the universal property of a free product, so $\pi_1(X)$ is the free product of the $\pi_1(V_i)$.

3.2 Let W be a union of (open) semicircles, one for each A_i, and all of them containing x_0. Let V_i be the union of W and A_i. Note that A_i is a deformation retract of V_i, so $\pi_1(V_i) = Z$, while W is contractible. Since $\pi_1(X)$ is a free product of infinite cyclic groups, it is a free group on the set that labels the circles.

2. As noted in class, the Hawaiian Earring is the union of infinitely many circles of decreasing diameter, all meeting at a point. For instance, for each positive integer n let S_n be a circle of radius $1/n$ in the plane, centered at $(1/n, 0)$, and let $X = \cup_n S_n$. (It helps to DRAW A PICTURE!) Let $x_0 = (0, 0)$. Show that this example does not meet the hypotheses of Exercise 3.2. (Again, already more or less done in class.)

Any neighborhood of the origin must contain a ball of radius ϵ around the origin, and therefore contains the entirety of all but finitely many circles, specifically all those with $n > 2/\epsilon$. In particular, there is no possible choice of W that would be contractible. Note also that under these circumstances a union of (open) semicircles would not be open, as the origin would be a limit point of the complement. This contradicts the description of the topology of X in Exercise 3.2

3. Not only does the Hawaiian Earring not meet the hypotheses of Exercise 3.2, it also doesn't meet the conclusions. Show that the inclusions $i_n : S_n \hookrightarrow X$ induce injections $\pi_1(S_n, x_0) \to \pi_1(X, x_0)$, and hence an injection $f : \prod^* \pi_1(S_n, x_0) \hookrightarrow \pi_1(X, x_0)$, but that f is not onto. In other words, $\pi_1(X, x_0)$ is not the free product of all of the cyclic groups $\pi_1(S_n, x_0)$. [Hint: Can every loop in X be expressed by a finite concatenation of loops in the S_n's?]

Consider the map $\rho_n : X \to S_n$ that is the identity on S_n and maps all points outside of S_n to the origin. This makes S_n a retract (although not a deformation retract) of X, so $\pi_1(S_n)$ injects in $\pi_1(X)$. By exercise III.4.3, this means that the induced map from the free product of the $\pi_1(S_n)$'s to $\pi_1(X)$ must be injective. However, consider a path that loops around S_1 in time $1/2$, then S_2 in time $1/4$, then S_3 in time $1/8$, etc. This path is continuous, as the limit as $t \to 1$ is simply the origin, but it involves going around an infinite number of loops, and is therefore not representable by a finite word, i.e., by an element of $\prod^* \pi_1(S_n)$.

4. Page 94, problem 3.4. Hint: with the correct choice of open sets, this is an easy corollary of problem 3.2.

For each integer i, let A_i be the set of points (x, y) such that either $y < 0$ or $y \geq 0$ and x is not an integer, or $y > 0$ and $x = i$. Let W be the set of points (x, y) such that either $y < 0$ or $y \geq 0$ and x is not an integer. These sets meet the conditions of problem 3.1, and each A_i deformation retracts to a circle, so $\pi_1(Y)$ is a product of infinite cyclic groups indexed by the integers, i.e., $\pi_1(Y)$ is a free group on a countable number of generators.

5. Page 95, problem 3.7. Let N_i be M_i with a disk cut out, and let $U_i \subset M_i$ be a slightly larger disk, so that $M_i = N_i \cup U_i$ and $N_i \cap U_i$ is homeomorphic to S^{n-1} times an interval. (Since M_1 and M_2 are manifolds, we can identify neighborhoods of a point with open sets in $\mathbb{R}^n$, so we know what it means to cut out a disk and take a bigger disk. All of this can be done explicitly with coordinates, if necessary.) Since U_i and $U_i \cap N_i$ are simply connected, $\pi_1(M_i)$ is isomorphic to $\pi_1(N_i)$. Next we write $M_1 \# M_2$ as the union of two sets, U and V, with U being N_1 (plus a little bit of N_2, using polar coordinates to glue them together) and V being the other way around. Since $U \cap V$ is simply connected (being homeomorphic to S^{n-1} times an interval), $\pi_1(U \cup V) = \pi_1(U) * \pi_1(V) = \pi_1(M_1) * \pi_1(M_2)$.

5. (Forgot how to count to 6) Page 103, problem 5.4.

Take an n-gon, orient all the edges counter-clockwise, and identify all of the edges. Since the boundary word is a^n, the fundamental group is $\langle a | a^n \rangle = Z_n$.

Lens spaces also work.

6. Let X_2 be a path-connected Hausdorff space, let $k > 2$, and let $f : S^{k-1} \to X_2$ be a continuous map. Let B be the closed unit ball in $\mathbb{R}^k$, and for each $x \in S^{k-1} \subset B$, identify x with $f(x) \in X_2$. Let X be the union of X_2 with B, modulo these identifications. This is called adjoining a k-cell to X_2. Let $x_0 \in X_2$. Show that $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X_2, x_0)$.

For $s < 1$, let $B_s \subset B$ be the closed ball of radius s around the origin. Let $U = B_{3/4}$ and let V be the complement of $B_{1/2}$ in X_2. U is contractible and $U \cap V$ deformation retracts to S^{k-1}, which is simply connected, so $\pi_1(X_2)$ is isomorphic to $\pi_1(V)$. But X_2 is a deformation retract of V, so $\pi_1(X_2)$ is isomorphic to $\pi_1(X_1)$.

7. Let X_1 be another path-connected Hausdorff space, and let X_2 be obtained by adjoining a 2-cell to X_1. Let $x_0 \in X_1$. Show that the inclusion $X_1 \hookrightarrow X_2$ induces a surjection $\pi_1(X_1, x_0) \to \pi_1(X_2, x_0)$. Give an example where this surjection is not an isomorphism.

The importance of exercises 6 and 7 is that it is possible to build any manifold, and a lot of other spaces, recursively. To get a space X, start with a set X_0 of disconnected points, called the 0-skeleton of X. Then add 1-cells to get a graph X_1 that is called the 1-skeleton. Then add 2-cells to X_1 to get X_2, add 3-cells to X_2 to get X_3, and in general add k-cells to the $k-1$ skeleton X_{k-1} to get the k-skeleton X_k. Finally, take $X = \cup_k X_k$. The upshot is that $\pi_1(X)$ is isomorphic to $\pi_1(X_2)$, and is a quotient of $\pi_1(X_1)$ by some relations that come from the 2-cells. This is both a blessing and a curse. The blessing is that we don't have to worry about higher dimensional structures when computing a fundamental group. The curse is that we can't use the fundamental group to keep track of higher dimensional structures. To do that we need other tools, such as homology.

In this instance, $\pi_1(U \cap V) = \pi_1(S^1) = Z$, so $\pi_1(X_2)$ is the quotient of $\pi_1(X_1)$ by the normal subgroup generated by the image of $\pi_1(U \cap V)$. Examples where this normal subgroup is nontrivial include all of the cases where X_2 is a compact surface and X_1 is the boundary of the polygon, with edges identified appropriately. An even simpler example is where X_1 is a circle and X_2 is a closed disk.