Algebraic Topology
Homework 7: Due Wednesday, October 20

Problem 1 *Back in early September, we proved that the fundamental group of a circle was infinite cyclic using Lebesgue numbers and an argument about angles. I want you to take that "keep track of angles" idea to its logical conclusion. Use the properties of the cover $R \to S^1$, $x \to e^{ix}$ and Lemmas 3.1, 3.2 and 3.3 to prove that $\pi_1(S^1) = Z$. Do not use Lebesgue numbers or partitions of the interval $[0,1]$ into pieces (except insofar as these concepts were required to prove Lemmas 3.1-3.3)*

Let $X = S^1$ with base point $1 + 0i$ and $\tilde{X} = \mathbb{R}$ with base point 0. By Lemma 3.1, any loop γ in S lifts to a path $\tilde{\gamma}$ in $\mathbb{R}$, and the endpoint of this path must be a multiple of 2π. If $\tilde{\gamma}(1) = 2\pi n$, consider the homotopy $\tilde{\gamma}_s(t) = s\tilde{\gamma}(t) + (1-s)2\pi nt$ between $\tilde{\gamma}$ and a uniform rotation. This projects to a homotopy between γ and a uniform rotation n times around the circle. By Lemma 3.3, homotopic paths must have lifts with the same endpoints, so paths with different values of n are not homotopic. Since every loop is homotopic to a uniform rotation, and since these rotations are characterized by how many times they go around, $\pi_1(S^1) = Z$.

Problem 2. *Let X be a path-connected and locally path-connected Hausdorff space, and consider the category C_1 whose objects are covering spaces of X, and whose morphisms are homomorphisms of covering spaces (as defined on page 130). Show that C_1 need not have a universal object. [Hint: it's sufficient to consider $X = S^1$.]*

Let $X = S^1$, whose covers are circles (n-fold covers) and the line (with infinitely many sheets). The n-fold covers can't be universal, since they don't map to the infinite cover. The infinite cover isn't universal, because it has n different maps to the n-fold cover (and infinitely many maps to itself). (If f is one such map, then $g_k(x) = f(x + 2\pi k)$ is another for every integer k.) A universal object has to have a *unique* morphism to every object, and this doesn't. The problem is that there's an ambiguity in where the base point goes. Once we resolve that ambiguity, in Problem 3, we do get universal properties.

Problem 3. *Now pick a base point $x \in X$, and consider the category C_2 whose objects are based covering spaces $(\tilde{X}, \tilde{x}, p)$ where $p(\tilde{x}) = x$, and whose morphisms are required to take base points to base points. Show that $(\tilde{X}, \tilde{x}, p)$ is a universal object if $\tilde{X}$ is simply-connected. (The converse is also true but is harder.)*

Let $(\tilde{X}', \tilde{x}', p')$ be another cover. A homomorphism $X \to X'$ is just a lift of the projection p to $\tilde{X}'$. This is guaranteed to exist by Theorem 5.1, since $p_*(\pi_1(\tilde{X}))$ is trivial. Once we specify that the lift must send $\tilde{x}$ to $\tilde{x}'$, it is unique by Lemma 3.2.

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The map r is clearly onto (since q is onto), and we just have to show that, for each point $z \in Z$ there exists a neighborhood U for which each path-component of $r^{-1}(U)$ maps homeomorphically to U. Pick U to be an elementary neighborhood of q, and let V be a path component of $r^{-1}(U)$. By Lemma 2.1, p restricted to $p^{-1}(V)$ is a cover of V. However, $p^{-1}(V) \subset q^{-1}(U)$, and each path-component of $q^{-1}(U)$ maps homeomorphically onto U. In particular, q is 1–1 on each path-component W of $p^{-1}(V)$, and so p is 1–1 on each path-component of $p^{-1}(V)$, so V can be considered an elementary neighborhood of p. W is homeomorphic to V and to $q(W) = r(V)$, so $r(V)$ is homeomorphic to V. This shows that r is a local homeomorphism.