Algebraic Topology
Homework 8 Solutions: Due Wednesday, October 27

Problem 1. Page 135, problem 7.2 – For example 2.4, you can restrict attention to the cover $\mathbb{R}^2 \to T^2$.

For example 2.2, our groups are $G = Z$ and $G_0 = nZ$. The normalizer of G_0 is all of G, and the automorphism group is the quotient Z_n, acting by rotation by $2\pi/n$.

For example 2.4, $\mathbb{R}^2 \to Z^2$, $G = Z^2$ and G_0 is trivial, so the automorphism group is Z^2, acting by translation in the obvious way.

For the first cover in example 2.7, $G = F_2$ and G_0 is the set of all words in a and b whose abelianization is zero, i.e. the commutator subgroup of $G = F_2$. This is a normal subgroup, so the automorphism group is $F_2/[F_2, F_2] = Z^2$, acting by translation in the obvious way.

For the second cover, G_0 is the set of all words whose abelianization is a power of b. This is again normal, and the quotient is Z, with generator a. The automorphism group just shifts things up by multiples of 3.

For example 2.8, $G = Z$, G_0 is trivial, and the automorphism group is Z, acting on $\mathbf{C}$ by adding multiples of $2\pi i$.

Example 2.9 is very similar to example 2.2. The automorphism group is Z_n, acting on $\mathbf{C} - 0$ by multiplication by powers of $\exp(2\pi i/n)$.

Problem 2. Page 144, problem 10.1

Let S_n be the orientable surface of genus n. Since S_n is a manifold, it has a universal cover, so its covers correspond to subgroups of $\pi_1(S_n)$. Consider the cover S whose fundamental group is $G_0 = [\pi_1(S_n), \pi_1(S_n)]$. This is a normal subgroup of $\pi_1(S_n)$, so the normalizer is all of $\pi_1(S_n)$ and the group of deck transformations is $G = \pi_1(S_n)/G_0$, which is the abelianization of $\pi_1(S_n)$, and is free abelian on $2n$ generators. Since G is infinite, S is non-compact and $S_n = S/G$. When $n = 1$, S is just the plane. When $n > 1$, S is a very complicated object.

Given a triangulated surface, we can order the vertices and describe edges and triangles by their vertices. If a triangle has three vertices v_i, v_j and v_k, and if $i < j < k$, we denote the triangle they span by t_{ijk}. Likewise e_{ij}, with $i < j$, is the edge from v_i to v_j. Let C_0 be the free abelian group whose generators are the vertices of the triangulation, let C_1 be the free abelian group whose generators are the edges, and let C_2 be the free abelian group whose generators are the triangles. In other words, a 2-chain is a formal sum of triangles, a 1-chain is a formal sum of edges, and a 0-chain is a formal sum of vertices. We define $\partial_2 t_{ijk} = e_{jk} - e_{ik} + e_{ij}$ and $\partial_1 e_{ij} = v_j - v_i$ and extend by linearity to all chains. It's easy to check that $\partial_1 \circ \partial_2 = 0$, so the homology of this chain complex is well defined, and is called the *simplicial homology* of the triangulated surface.

Problem 3 *Revisit problems 7.2, 7.3, and 7.4 on page 19, only now compute the simplicial homology of each triangulation.*

This is grungy, but it goes through. (MATLAB helps with the linear algebra, by the way.) For 7.2, we have $C_0 = \mathbf{Z}^4$, with basis v_1, v_2, v_3, v_4, $C_1 = \mathbf{Z}^6$ with basis $e_{12}, e_{13}, e_{14}, e_{23}, e_{24}, e_{34}$ and $C_2 = \mathbf{Z}^4$ with basis $t_{123}, t_{124}, t_{134}, t_{234}$. The boundary maps are $\partial_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$ and $\partial_1 = \begin{pmatrix} -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$. The kernel of ∂_2 is the span of $t_{123} - t_{124} + t_{134} - t_{234}$, so $H^2 = Z$. The kernel of ∂_1 is the image of ∂_2, so $H^1 = 0$. $H^0 = Z$, where every vertex is identified with every other vertex.

For 7.3, we list the 10 triangles and 15 edges and 6 vertices in numerical order to get bases for $C_2 = \mathbf{Z}^{10}$, $C_1 = \mathbf{Z}^{15}$ and $C_0 = \mathbf{Z}^6$. The boundary maps are

$$\partial_2 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

and

$$\partial_1 = \begin{pmatrix} -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix},$$

which have rank 10 and 5, respectively. However, the image of ∂_2 is an index-2 subgroup of the kernel of ∂_1, so $H^2 = 0$, $H^1 = \mathbf{Z}_2$, and $H^0 = \mathbf{Z}$. (This is RP^2, by the way)

For 7.4, we have $C_2 = \mathbf{Z}^{14}$, with basis t_{124}, t_{126}, t_{134}, t_{137}, t_{156}, t_{157}, t_{235}, t_{237}, t_{245}, t_{267}, t_{346}, t_{356}, t_{457}, and t_{467}. $C_1 = \mathbf{Z}^{21}$, with basis $e_{12}, e_{13}, \ldots, e_{17}, e_{23}, \ldots, e_{57}, e_{67}$, and

$C_0 = \mathbf{Z}^7$. The boundary matrices are

$$\partial_2 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and $\partial_1 =$

$$\begin{pmatrix} -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

Now ∂_2 has rank 13, with pivots in the first 13 columns, with kernel spanned by $(1, -1, -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, 1, -1,)^T$, so $H^2 = \mathbf{Z}$, and the image of ∂_2 is spanned by the first 13 columns of ∂_2. The reduced row-echelon form of ∂_1 has rank 6 and pivots in the first 6 columns, so the last 15 variables of a vector in the kernel of ∂_1 determine the first 6. We therefore look at the last 15 entries of the first 13 columns of ∂_2. Column-reducing this matrix (over the integers) gives a matrix with pivots in all but the 12th and 15th rows, meaning that the image of ∂_2 is a subspace of the kernel of ∂_1 of codimension 2, and $H^1 = \mathbf{Z}^2$. The fact that the pivots are all 1 shows that there is no torsion. As usual $H^0 = \mathbf{Z}$.

We now generalize to higher dimensions. Working in $\mathbb{R}^k$, the standard k-simplex is the convex hull of the points $v_0 = (0, 0, \ldots, 0)$, $v_1 = (1, 0, 0, \ldots, 0)$, $v_2 = (0, 1, 0, \ldots, 0)$, *through* $v_k = (0, 0, \ldots, 0, 1)$. *We will label this is* Δ_0^k. *Now let* A *be a rank-*k *affine map from* $\mathbb{R}^k$ *to* $\mathbb{R}^n$ *that takes* $v_0, \ldots, v_k$ *to* $k+1$ *distinct points* $w_0, \ldots, w_k$. *The image of* Δ_0^k *under this map is called a* k*-simplex with vertices* $w_0, \ldots, w_k$ *and is denoted* $\Delta_{w_0,\ldots,w_k}$. *We order the vertices in any simplicial complex, and generally list the vertices of each simplex in increasing order.*

(Actually, it's sometimes useful to write the vertices in arbitrary order, with the understanding that t_{jik} *means* $-t_{ijk}$*, etc. That is, permuting the order of the vertices means multiplying by the sign of the permutation. For instance,* $\partial t_{ijk} = e_{jk} - e_{ik} + e_{ij}$*, regardless of the ordering of* i*,* j*, and* k*. Still, we usually take as a basis for* C^k *the* k*-simplices with vertices written in increasing order.)*

The boundary of a simplex $\Delta_{w_0,\ldots,w_k}$*, is defined to be* $\sum_{i=0}^k (-1)^i \Delta_{w_0,\ldots,w_{i-1},w_{i+1},\ldots w_k}$. *The* $k-1$ *simplex based on all the vertices except* w_i *is called the* i*-th face of the* k*-simplex.*

Problem 4. Show that $\partial_{k-1} \circ \partial_k = 0$. Also show that the definition of $\partial \Delta_{w_0,\ldots,w_k}$ works even if the vertices are NOT listed in increasing order.

Note that $\partial_{k-1}\partial_k \Delta_{w_0,\ldots,w_k}$ involves faces with two vertices removed (say, w_i and w_j, with $i < j$), and this can occur by removing the ith vertex and then the jth, or by removing the jth vertex and then the ith. The first gives a sign of $(-1)^{i+j-1}$ while the second gives a sign of $(-1)^{i+j}$, so the sum is zero. This is qualitatively the same argument that we used to show that $\partial^2 = 0$ for cubes.

To show that the definitions work with the order of vertices scrambled, we only need to show that transposing two adjacent vertices preserves the definition, in that any permutation is a product of such transpositions.

The simplex $\Delta_1 = \Delta_{w_0,\ldots,w_{i-1},w_{i+1},w_i,w_{i+2},\ldots}$ is minus $\Delta_2 = \Delta_{w_0,\ldots,w_i,w_{i+1},\ldots}$. For j not equal to i or $i+1$, the j-th face Δ_1 is minus the j-th face of Δ_2, since they differ by transposition of w_i and w_{i+1}. The ith face of Δ_1 is the $i+1$st face of Δ_2, and the $1+1$st face of Δ_1 is the ith face of Δ_2. This means that the boundary of Δ_2 is minus the boundary of Δ_1, as it should be.

A simplicial complex is a collection of simplices such that (1) every collection of vertices determines at most one simplex (for instance, there can't be two edges from v_i *to* v_j*), and (2) the intersection of any two simplices is the simplex defined by their common vertices (or is empty if they have no vertices in common), and (3) if a simplex is in the collection, then so are all its faces, and the faces of the faces, etc. (e.g., if* t_{ijk} *is a 2-simplex in the collection, then* e_{ij}*,* e_{ik} *and* e_{jk} *are all 1-simplices in the collection, and* v_i*,* v_j *and* v_k *are all 0-simplices.) The simplest simplicial complex of dimension* k *consists of a single* k*-simplex, together with its faces and sub-faces.*

Problem 5. Compute the homology of the 2-simplex Δ_0^2 (together with its faces and sub-faces, of course). Repeat for Δ_0^3. Repeat for the complex obtained from the faces of Δ_0^3 without the 3-simplex itself (i.e., the surface of a tetrahedron).

For Δ_0^2, we have $C_2 = \mathbf{Z}$, $C_1 = \mathbf{Z}^3$ (with basis e_{12}, e_{13}, e_{23}) and $C_0 = \mathbf{Z}^3$. The boundary maps are $\partial_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ and $\partial_1 = \begin{pmatrix} -1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$, the kernel of ∂_1 equals the image of ∂_2, and we have $H_2 = H_1 = 0$, $H_0 = \mathbf{Z}$. For Δ_0^3, we have $C_3 = \mathbf{Z}$, $C_2 = \mathbf{Z}^4$, $C_1 = \mathbf{Z}^6$ and $C_0 = \mathbf{Z}^4$, with $\partial_3 = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}$, and with ∂_2 and ∂_1 exactly as in problem 7.2 of page 19, as computed above. Since the image of ∂_3 equals the kernel of ∂_2, we have $H_3 = H_2 = H_1 = 0$, $H_0 = \mathbf{Z}$. Finally, the faces of Δ_0^3 without the interior is *exactly* the complex of 7.2, for which we computed $H_2 = H_0 = \mathbf{Z}$, $H_1 = 0$.