Algebraic Topology
Homework 9: Due Wednesday, November 3

Problem 1. Page 163, problem 2.1

The rational numbers with their usual topology and a countable discrete set are the same when it comes to homology, since in either case there are countably many path-components, each of which is a point, so by Proposition 2.7 (or Corollary 2.5) H_0 is a free abelian group on countably many generators, and H_k is trivial for $k > 0$. The moral of the story is that homology treats path components one at a time, no matter how bunched up they are.

Problem 2. Page 166, problem 3.3. The hard part is the last statement.

Since $r \circ i$ is the identity on A, $r_* \circ i_*$ is the identity on $H_k(A)$, so i_* is one-to-one and r_* is onto. Now the image of any homomorphism of groups is isomorphic to the source mod the kernel, so $H_k(A)$ is isomorphic to $H_k(X)$ mod the kernel of r_*. That is, we have a short exact sequence

$$0 \to ker(r_*) \hookrightarrow H_k(X) \to H_k(A) \to 0,$$

with the second nontrivial map being r_*. This sequence splits, since we have an injection of $H_k(A)$ into $H_k(X)$ (namely i_*) such that $r_* \circ i_*$ is the identity. This implies that $H_k(X)$ is the direct sum of $i_*(H_k(A))$ and the kernel of r_*.

Cone operators. Let X be a star-like subset of Euclidean space, meaning that if $x \in X$, then $tx \in X$ for every $t \in [0, 1]$.

As usual, let $C_n(X)$ be generated by the singular n-chains on X, modulo the degenerate chains. Let $C_{-1} = \mathbf{Z}$, with the usual augmentation map $C_0 \to C_{-1}$ that counts points. That is, we're computing *reduced* homology. Consider the *cone operator* $\phi : C_n \to C_{n+1}$ defined as follows. If $n \geq 0$ and T is an n-cube, let $\phi(T)(x_1, \ldots, x_{n+1}) = x_1 T(x_2, \ldots, x_{n+1})$. If $n = -1$, let $\phi(1)$ be the 0-cube that maps to the origin.

Problem 3. Show that $\partial \circ \phi + \phi \circ \partial$ is the identity on C_n. Use this fact to prove that the reduced homology of X is trivial.

First suppose that T is an n-cube with $n > 0$. Consider the faces of $\phi(T)$. The front 1-face A_1 is degenerate, the back 1-face B_1 is T itself, and the remaining front and back faces are cones of the corresponding faces of T, albeit with indices shifted by one. That is, $A_k(\phi(T)) = \phi(A_{k-1}(T))$ and $B_k(\phi(T)) = \phi(B_{k-1}(T))$. This shows that $\partial(\phi(T)) + \phi(\partial T) = T$.

We have to be a little more careful in dimension zero, since ∂_0 is the augmentation map that counts points. If T is a 0-cell, i.e. a point p, then $\phi(T)$ is a path from the origin to p, and $\partial\phi(T)$ is 1 times p plus -1 times the origin (viewed as a 0-cell). However, $\partial(T) = 1$, so $\phi(\partial(T))$ equals the origin, and $\partial(\phi(T)) + \phi(\partial T) = T$.

In dimension -1, $C_{-1} = \mathbf{Z}$ is generated by the number 1, and $\phi(\partial(1)) = 0$, and $\partial\phi(T) = \partial\{0\} = 1$. So in all dimensions, $\partial\phi + \phi\partial$ is the identity operator, hence a homotopy operator between the identity map and the zero map. This shows that the identity equals zero on (reduced) homology, hence that the reduced homology is trivial in all dimensions. (The regular homology is trivial in all positive dimensions, and $H_0 = \mathbf{Z}$.)

Problem 4. Let Y be any topological space, and let X be the cone of Y. Modify the construction of Problem 3 to show that the reduced homology of X is trivial.

Let the cone of Y be $[0,1] \times Y/\sim$, where $(0, y_1) \sim (0, y_2)$ for any $y_1, y_2 \in Y$. If T is a k-cube on Y, suppose that $T(x_1, \ldots, x_n) = (t(x_1, \ldots, x_n), T_0(x_1, \ldots, x_n))$, where $T_0 : [0,1]^n \to X$ is only defined if $t(x_1, \ldots, x_n) \neq 0$. Now let $\phi(T)(x_1, \ldots, x_{n+1}) = (x_1 t(x_2, \ldots, x_{n+1}), T_0(x_2, \ldots, x_{n+1}))$. For -1-chains, let $\phi(1)$ be the cone point.

Exactly as before, the front 1-face of $\phi(T)$ is degenerate (except in dimension zero), the back 1-face is T itself, the front k-face for $k > 1$ is ϕ of the front $k-1$ face of T, and the back k-face for $n > 1$ is ϕ of the back $k-1$ face of T, and $\partial\phi + \phi\partial$ is the identity, and the rest of the calculation follows verbatim from problem 3.

Problem 5. More generally, let X be any contractible space. Use an appropriate cone operator to show that the reduced homology of X is trivial.

Please do problems 3–5 in order. Don't just do problem 5 and then say "3 and 4 are special cases". They are, but the idea is to go from the concrete and specific to the abstract and general in stages.

Let $F : [0,1] \times X \to X$ be a homotopy between a constant map f_0 and the identity map f_1. Let $\phi(T)(x_1, \ldots, x_{n+1}) = F(x_1, T(x_2, \ldots, x_{n+1}))$. For -1 chains, let $\phi(1)$ be the image of f_0.

Again, exactly as before, the front 1-face of $\phi(T)$ is degenerate (as usual, except in dimension zero, where the augmentation saves the day), the back 1-face is T itself, the front k-face for $k > 1$ is ϕ of the front $k-1$ face of T, and the back k-face for $n > 1$ is ϕ of the back $k-1$ face of T, and $\partial\phi + \phi\partial$ is the identity.

Problem 6. As a continuation of last week's assignment on simplicial homology, see if you can generalize, finding (and proving!) a formula for the homology of Δ_0^k (including its faces and sub-faces). The key is constructing an appropriate cone operator, albeit in the category of simplicial complexes and simplicial chains rather that topological spaces and singular chains.

The homology is trivial, except for H^0, which is $\mathbf{Z}$. As with problems 3–5, the key to proving this is finding a *cone operator* $\phi : C_n \to C_{n+1}$ such that, for each $n > 0$, the identity on C_n can be written as $\partial \circ \phi + \phi \circ \partial$. This operator is defined as follows:

If $\{w_0, \ldots, w_n\}$ are written in increasing order, then $\phi(\Delta_{w_0,\ldots,w_n}) = \Delta_{0,w_0,\ldots,w_n}$ if $w_0 \neq 0$, and is zero if $w_0 = 0$. If $w_0 \neq 0$, then the 0th face of $\partial\phi(\Delta_{w_0,\ldots,w_n})$ is just $\Delta_{w_0,\ldots,w_n}$ itself, while the ith face for $i > 0$ is ϕ of the $i-1$st face of $\Delta_{w_0,\ldots,w_n}$. If $w_0 = 0$, then $\partial\phi\Delta = 0$, but $\phi(\partial(\Delta)) = \Delta$, thanks to the 0-th face of Δ.

The argument breaks down at dimension zero, since if α is the obvious 0-cube, $\partial\phi(\alpha)$ is zero, as is $\phi(\partial\alpha)$. H_0 is not zero, but is $\mathbf{Z}$, since Δ_0^k is path-connected.