Algebraic Topology Midterm Exam Solutions, October 15, 2010

1. (30 points) (a) Let X be a punctured torus and let γ be a loop around the puncture, as depicted on the blackboard. Let x_0 be a point on γ. Compute $\pi_1(X, x_0)$ and express the class of γ in terms of the generators and relations of that group.

This was essentially a homework problem (not to mention part of last year's exam). X retracts to the wedge of two circles, so $\pi_1(X, x_0) = F_2 = \langle a, b\rangle$ and $[\gamma]$ is the boundary word $aba^{-1}b^{-1}$.

(b) Now let Y be the union of X with a disk, as shown on the board. Compute $\pi_1(Y, x_0)$ and compute the class of γ in $\pi_1(Y, x_0)$.

Let D be the disk and let V be a neighborhood of the disk, and let U be a neighborhood of the complement of D. V deformation retracts to D, and hence to a points, while U deformation retracts to X. By van Kampen, since V is contractible, $\pi_1(Y)$ is the quotient of $\pi_1(U)$ by the image of $\pi_1(U \cap V)$ in $\pi_1(U)$. Since $U \cap V$ retracts to a circle going around the torus once, (call that b), $\pi_1(Y, x_0) = \langle a, b|b\rangle = \langle a\rangle = \mathbb{Z}$. The class of γ is now the image of $aba^{-1}b^{-1}$ when we set $b = 1$, namely $[\gamma] = 1$.

(c) Now let Z be the union of $T^2\#T^2$ and two disks, as shown on the board. Compute $\pi_1(Z)$.

Again we use van Kampen, now with U and V being the left and right halves of the picture (overlapping slightly, of course). U and V are both homeomorphic to Y, so $\pi_1(U) = \pi_1(V) = \mathbb{Z}$, while $U \cap V$ is an annulus, with $\pi_1(U \cap V) = \mathbb{Z}$. The generator of $\pi_1(U \cap V)$ is the class of γ, which maps to the identity in both $\pi_1(U)$ and $\pi_1(V)$, so van Kampen says that $\pi_1(Z) = \mathbb{Z} * \mathbb{Z} = F_2$. Note that the image of γ really matters in this calculation!

2. (20 points) The infinite dihedral group D_∞ is a group of symmetries of the real line generated by a translation $t(x) = x + 1$ and a reflection $r(x) = -x$. Show that D_∞ is isomorphic to the free product $\mathbb{Z}_2 * \mathbb{Z}_2$.

Let a, b be the generators of $\mathbb{Z}_2 * \mathbb{Z}_2$, with $a^2 = b^2 = 1$. We map $\mathbb{Z}_2 * \mathbb{Z}_2$ to D_∞ by $a \to r$ (that is $a(x) = -x$) and $b \to tr$ (that is, $b(x) = 1 - x$.). This is well-defined since a and b both go to elements of order 2, so both define maps $\mathbb{Z}_2 \to D_\infty$, and, by the universal property of free products, any two maps from $\mathbb{Z}_2 \to D_\infty$ induce a (unique) map $\mathbb{Z}_2 * \mathbb{Z}_2 \to D_\infty$. I claim this map is an isomorphism.

It's certainly onto, since the image of ba is t and the image of a is r. We must show that it's 1–1. Note that the image of ab is t^{-1}. We showed in homework that all elements of $\mathbb{Z}_2 * \mathbb{Z}_2$ are of the form $(ab)^n$, $(ab)^n a$, $(ba)^n$ or $(ba)^n b$ with $n \geq 0$, as these are all the reduced words. These four classes map to t^{-n}, $t^{-n} r = r t^n$, t^n and $t^{n+1} r = r t^{-(n+1)}$, which are all distinct (except for t^0 and t^{-0}, which come from $(ab)^0$ and $(ba)^0$, which are of course the same). QED.