L' Hôpital = L'Hospital

L'-oh-pee-<u>tal</u>

$$\lim_{x \to 1} \frac{\ln(x)}{x-1}$$

Indeterminate Form "$\frac{0}{0}$"

$$\lim_{x \to \infty} \frac{\ln(x)}{x-1}$$

$$\lim_{x \to \infty} \frac{x^2}{e^x} \quad \text{or} \quad \lim_{x \to \infty} \frac{x^n}{e^x}$$

$$\lim_{x \to 0^+} x \ln(x)$$

$$\lim_{x \to \pi/2}$$

$$\lim_{x\to 1} \frac{\ln(x)}{x-1} = \lim_{x\to 1} \frac{\ln(x)-\ln(1)}{x-1} \overset{\text{def}}{=\!=} \frac{d}{dx}\ln(x)\Big|_{x=1}$$

$$= \left(\frac{1}{x}\right)\Big|_{x=1} = \frac{1}{1} = 1$$

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \text{"}\frac{0}{0}\text{"}$$

Suppose $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$

Suppose f and g are differentiable near a

Suppose $g'(x) \neq 0$ for all x near a (except $x=a$)

Suppose $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists.

Then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

Ex: $\lim_{x\to 0}$ ~~cos sin(x)~~ $\frac{\cos(x)-\cos(2x)}{x+x^2}$

$$= \lim_{x\to 0} \frac{-\sin(x)-(-2\sin(2x))}{1+2x} = \frac{0}{1} = 0$$

Extensions of L'Hôpital's rule

1) Works for 1-sided limits ($x \to a^+$ or $x \to a^-$)

2) Works for $\lim_{x \to \pm\infty}$

3) Works for $\frac{\infty}{\infty}$

$$\lim_{x \to \infty} \frac{\ln(x)}{x-1} = \lim_{x \to \infty} \frac{(1/x)}{1} = \frac{0}{1}$$

$$\lim_{x \to \infty} \frac{e^x}{x^2} = \lim_{x \to \infty} \frac{e^x}{2x} = \lim_{x \to \infty} \frac{e^x}{2} = +\infty$$

$$\lim_{x \to \infty} \frac{e^x}{x^{137}} = \lim \frac{e^x}{137x^{136}} = \cdots = \lim \frac{e^x}{(137)!} = \infty$$

$$\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln(x)}{1/x} \quad \text{“}0 \cdot \infty\text{”}$$

$$= \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = \lim_{x \to 0^+} (-x) = 0$$

$$\lim_{x \to \frac{\pi}{2}^-} (\sec(x) - \tan(x)) \qquad \text{"}\infty - \infty\text{"}$$

$$= \lim_{x \to \frac{\pi}{2}^-} \left(\frac{1}{\cos(x)} - \frac{\sin(x)}{\cos(x)} \right) = \lim_{x \to \frac{\pi}{2}^-} \left(\frac{1 - \sin(x)}{\cos(x)} \right)$$

$$= \lim_{x \to \frac{\pi}{2}^-} \left(\frac{-\cos(x)}{-\sin(x)} \right) = \frac{0}{-1} = 0$$

$$\lim_{x \to 0} x^x \qquad y = x^{x} \qquad 0^0$$

$$\ln y = x \ln x \quad ; \quad y = e^{x \ln x}$$

$$\lim_{x \to 0^+} (\ln y) = \ln (\lim_{x \to 0^+} y) = \lim_{x \to 0^+} x \ln(x) = 0$$

$$y \Rightarrow 1$$

$$\lim y = e^{\lim (\ln y)} = e^0 = 1$$

Why does L'Hôpital's rule work?

Simplest case: $\frac{0}{0}$, ~~a fm~~ & $g'(a) \neq 0$

$$\lim_{x\to a} \frac{f(x)}{g(x)} \qquad \text{where } f(a)=g(a)=0,\ g'(a)\neq 0.$$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim \left(\frac{f(x)-f(a)}{g(x)-g(a)}\right)$$

$$= \lim_{x\to a} \left(\frac{(f(x)-f(a))/(x-a)}{(g(x)-g(a))/(x-a)}\right)$$

$$= \frac{\lim_{x\to a} (f(x)-f(a))/(x-a)}{\lim_{x\to a} (g(x)-g(a))/(x-a)} = \frac{f'(a)}{g'(a)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

(If $g'(a)=0$, need MVT to make argument work)

$f(x) \to \infty$
$g(x) \to \infty$

$$L = \lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{1/g(x)}{1/f(x)} \text{ “} = \frac{0}{0} \text{”}$$

$$= \lim \frac{-g'(x)/g^2}{-f'(x)/f^2} = \lim \left(\frac{f^2}{g^2}\right) \Big/ \lim \left(\frac{f'}{g'}\right)$$

$$= L^2 \Big/ \lim \left(\frac{f'}{g'}\right)$$

$$\lim f'/g' = L$$

$$L = \lim_{x \to \infty} \frac{f(x)}{g(x)} \qquad t = \frac{1}{x}$$

$$= \lim_{t \to 0^+} \frac{f(1/t)}{g(1/t)} = \lim_{t \to 0^+} \frac{f'(1/t) \cdot (-1/t^2)}{g'(1/t) \cdot (-1/t^2)}$$

$$= \lim_{t \to 0^+} \frac{f'(1/t)}{g'(1/t)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)}$$

~~When~~ To handle $0 \cdot \infty$.

1) Rewrite $f(x) \cdot g(x)$ as $\dfrac{f(x)}{1/g(x)} \swarrow \frac{0}{0}$ or $\dfrac{g(x)}{1/f(x)} \swarrow \frac{\infty}{\infty}$

2) Can apply L'Hôpital recursively.

3) To handle $\infty - \infty$, use algebra to turn it into $\frac{\infty}{\infty}$ or $\frac{0}{0}$

4) To handle 0^0 or ∞^0 or 1^∞ $\left(\lim_{x \to \infty} \left(1+\frac{n}{x}\right)^x\right)$

$$y = f(x)^{g(x)}$$

$$\ln y = g(x) \ln(f(x)) = 0 \cdot \infty \text{ or } \infty \cdot 0$$

$$\lim y = e^{\lim (\ln y)}$$

$$\lim_{x \to 0} \frac{\sin(2x) \quad \cancel{\cos(x) - \cos(2x)}}{1+x} = \frac{0}{1} = 0$$

L'Hôpital does NOT apply to $\frac{0}{1}$!