

# How to Take Derivatives

## 1 Know your basic building blocks

Almost all functions that you will encounter are built from the following functions:  $x^n$ ,  $e^x$ ,  $\ln(x)$ ,  $\sin(x)$  and  $\cos(x)$ . Memorize the following table until it's as familiar to you as the 1-digit multiplication table that your memorized in elementary school.

$f(x)$	$f'(x)$
$x^n$	$nx^{n-1}$
$e^x$	$e^x$
$\ln(x)$	$1/x$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$

These formulas can all be derived directly from the definition of the derivative, with the except of  $\ln(x)$ , which requires a little extra work.

## 2 Advanced building blocks

Either memorize the derivatives of the following functions, or know how to derive them from the derivatives of the basic building blocks. Either way, get to the point where it takes you a minute, at most, to figure out the derivatives of these functions.

$f(x)$	$f'(x)$
$a^x$	$a^x \ln(a)$
$\log_a(x)$	$\frac{1}{x \ln(a)}$
$\tan(x)$	$\sec^2(x)$
$\cot(x)$	$-\csc^2(x)$
$\sec(x)$	$\sec(x) \tan(x)$
$\csc(x)$	$-\csc(x) \cot(x)$
$\sin^{-1}(x)$	$1/\sqrt{1-x^2}$
$\tan^{-1}(x)$	$1/(1+x^2)$
$\sec^{-1}(x)$	$\frac{1}{x\sqrt{x^2-1}}$

### 3 Product and quotient rules

The product and quotient rules are:

$$\frac{d}{dx} f(x)g(x) = f(x)g'(x) + f'(x)g(x)$$

$$\frac{d}{dx} \frac{f(x)}{g(x)} = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$

These are sometimes expressed in terms of “ $u$ ” and “ $v$ ”, as

$$d(uv)/dx = u(dv/dx) + (du/dx)v$$

$$\frac{d}{dx} \left( \frac{u}{v} \right) = \frac{v(du/dx) - u(dv/dx)}{v^2}$$

Notice that the  $v(du/dx)$  term is positive and the  $u(dv/dx)$  term is negative. The way to remember that is that, if  $u$  and  $v$  are both positive, then increasing the numerator  $u$  will increase the ratio  $u/v$ , while increasing the denominator  $v$  will decrease the ratio  $u/v$ .

For example, to take the derivative of  $x^2 \sin(x)$ , let  $u = x^2$  and  $v = \sin(x)$ . Then  $u' = 2x$  and  $v' = \cos(x)$ , so the derivative of  $uv$  is  $x^2 \cos(x) + 2x \sin(x)$ .

Likewise, to take the derivative of  $\tan(x) = \sin(x)/\cos(x)$ , let  $u = \sin(x)$  and  $v = \cos(x)$ , so  $u' = \cos(x)$  and  $v' = -\sin(x)$ , and  $d(u/v)/dx = (vu' - uv')/v^2 = \frac{\cos(x)\cos(x) - (\sin(x))(-\sin(x))}{\cos^2(x)}$ , which simplifies to  $1/\cos^2(x) = \sec^2(x)$ , since  $\sin^2(x) + \cos^2(x) = 1$ .

### 4 Chain Rule

The chain rule allows you to take the derivative of compound functions like  $\sin(x^2)$  or  $(\sin(x))^2$ . If  $f(x) = \sin(x)$  and  $g(x) = x^2$ , then  $\sin(x^2) = f(g(x))$  and  $(\sin(x))^2 = g(f(x))$ . The rule says:

$$(f(g(x)))' = f'(g(x)) \cdot g'(x),$$

so the derivative of  $\sin(x^2)$  is  $\cos(x^2) \cdot 2x$ , or  $2x \cos(x^2)$ , while the derivative of  $\sin^2(x)$  is  $2 \sin(x) \cdot \cos(x)$ .

It's often useful to let  $u = g(x)$ , so our rule becomes

$$\frac{df(u)}{dx} = f'(u) \frac{du}{dx}.$$

Combining this with our derivatives of basic functions, we get:

$f(x)$	$f'(x)$
$u^n$	$nu^{n-1} \cdot du/dx$
$e^u$	$e^u \cdot du/dx$
$\ln(u)$	$(du/dx)/u$
$\sin(u)$	$\cos(u) \cdot (du/dx)$
$\cos(u)$	$-\sin(u) \cdot (du/dx)$

In particular, when taking the derivative of  $\sin(x^2)$ , just let  $u = x^2$ , so we get the derivative of  $\sin(u)$  being  $\cos(u) \cdot 2x = 2x \cos(x^2)$ . When taking the derivative of  $\sin^2(x)$ , take  $u = \sin(x)$ , so the derivative of  $u^2$  is  $2u(du/dx) = 2\sin(x) \cos(x)$ .

Yet another form of the chain rule comes from taking  $y = f(u)$ , where  $u = g(x)$ , so we have

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}.$$

$dy/du$  is another name for  $f'(u) = f'(g(x))$ , while  $du/dx$  is another name for  $g'(x)$ .

## 5 Combining Rules

Many functions can't be cracked open with a single rule. Instead, use a rule to break the problem down into (possibly several) simpler problems. Then use another rule on each of those. Repeat as long as needed to get your answer. For instance, suppose that  $F(x) = \frac{\sin(x^2)}{1+e^{2x}}$ . This is a ratio, so

$$F'(x) = \frac{(1 + e^{2x}) \frac{d(\sin(x^2))}{dx} - \sin(x^2) \frac{d(1+e^{2x})}{dx}}{(1 + e^{2x})^2},$$

by the quotient rule. But that still leaves us with the question of how to compute the derivatives of  $\sin(x^2)$  and  $(1 + e^{2x})$ . Each of those can be computed using the chain rule.

**It's very dangerous to combine steps!** Until you've really got the hang of it, I strongly recommend writing out your calculations with at most

one rule per line. For instance, you might write:

$$\begin{aligned}
 F'(x) &= \frac{(1+e^{2x})\frac{d(\sin(x^2))}{dx} - \sin(x^2)\frac{d(1+e^{2x})}{dx}}{(1+e^{2x})^2} && \text{by the quotient rule} \\
 &= \frac{(1+e^{2x})(\cos(x^2))(2x) - \sin(x^2)\frac{d(1+e^{2x})}{dx}}{(1+e^{2x})^2} && \text{by the chain rule applied to } \sin(x^2) \\
 &= \frac{(1+e^{2x})(\cos(x^2))(2x) - \sin(x^2)e^{2x}(d(2x)/dx)}{(1+e^{2x})^2} && \text{by the chain rule applied to } e^{2x} \\
 &= \frac{2x(1+e^{2x})\cos(x^2) - 2\sin(x^2)e^{2x}}{(1+e^{2x})^2} && \text{by algebra.}
 \end{aligned}$$

**Eventually** you'll get the hang of it to the point that you can do several operations in one line, but please be patient.

There's not always an obvious order in which you apply the rules. Think about the structure of the function. Would you describe it as a product of two simpler functions? If so, apply the product rule first! Would you describe it as a quotient? If so, apply the quotient rule first. Is it instead a power, or a log, or an exponential, or a trig function of some complicated expression (which may itself involve products, quotients, or further nesting)? If so, apply the chain rule first.

## 6 Implicit differentiation

Derivatives aren't just for functions that you already know a formula for. You can meaningfully ask for the rate of change of anything! In particular, if a certain equation holds for all values of  $x$ , then the derivative of the left hand side must equal the derivative of the right hand side. If the expressions involve  $y$ , then the derivatives will involve  $y' = dy/dx$ , thanks to the chain rule. Then solve for  $dy/dx$  by putting all the terms that include  $dy/dx$  on one side of the equation, and all the terms that don't on the other side.

For instance, if  $x^2y + y^3 = \sin(x)$ , then we would get  $2xy + x^2y' + 3y^2y' = \cos(x)$ . After grouping, we'd have  $(x^2 + 3y^2)y' = \cos(x) - 2xy$ , or  $y' = \frac{\cos(x) - 2xy}{x^2 + 3y^2}$ . Note that the answer is typically an expression involving both  $x$  and  $y$ . Occasionally we can simplify this into something that just involves  $x$ , but usually we can't.

There are 2 main uses of implicit differentiation. The first is to get information about curves where  $x$  and  $y$  are related in a way that's more complicated than just  $y = f(x)$ . At each point  $(x, y)$  on the curve, we can

figure out the derivative, plot the tangent line, and estimate what the curve is doing nearby.

The second use is to compute the derivatives of inverse functions. If  $y = f^{-1}(x)$ , then  $x = f(y)$ , so  $1 = f'(y)y'$ , so  $y' = 1/f'(y)$ . Often that can be expressed in terms of  $x$ .

For instance, to compute the derivative of  $\tan^{-1}(x)$ , we write:

$$\begin{aligned}y &= \tan^{-1}(x) \\x &= \tan(y) \\1 &= \sec^2(y)y' \\y' &= 1/\sec^2(y) \\y' &= 1/(1 + \tan^2(y)) \\y' &= 1/(1 + x^2)\end{aligned}$$

The derivatives of  $\ln(x)$ ,  $\sin^{-1}(x)$  and  $\sec^{-1}(x)$  can be derived similarly.

Another way to say this is that, since  $x = f(y)$ ,  $dx/dy = f'(y)$ . However,  $dy/dx = 1/(dx/dy) = 1/f'(y)$ . In other words,  $dx/dy$  is the reciprocal of  $dy/dx$ .

## 7 Logarithmic derivatives

If sometimes happens that it's easier to take the derivative of  $\ln(y)$  than of  $y$ . In those cases, we can get  $y'$  indirectly, as follows:

By the chain rule,

$$\begin{aligned}\frac{d(\ln(y))}{dx} &= \frac{dy/dx}{y}, \text{ or equivalently} \\ \frac{dy}{dx} &= y \frac{d\ln(y)}{dx}.\end{aligned}$$

So, to compute  $dy/dx$ , first compute  $\ln(y)$ , then take its derivative, and then multiply the answer by  $y$ . This procedure isn't always helpful, and in some cases it can make your computation much harder, but in other cases it can make your computation much easier.

For instance, suppose that  $y = x^x$ . You can't take the derivative of  $x^x$  from the product, quotient or chain rules (at least not without some serious tricks), so it looks like we're stuck. However,  $\ln(y) = x \ln(x)$ , which is a function that we **do** know how to differentiate. Thanks to the product rule,

the derivative of  $x \ln(x)$  is  $x(1/x) + (1)(\ln(x)) = 1 + \ln(x)$ , so the derivative of  $x^x$  is  $x^x(1 + \ln(x))$ .

So when should you use logarithmic derivatives? *Whenever  $\ln(y)$  is easier to differentiate than  $y$ .* If  $y$  involves a bunch of powers, products and quotients, then  $\ln(y)$  is likely to be simpler, thanks to the three basic rules of logs:

$$\ln(ab) = \ln(a) + \ln(b), \quad \ln(a/b) = \ln(a) - \ln(b), \quad \ln(a^r) = r \ln(a).$$

On the other hand, if  $y$  is a sum of terms, then  $\ln(y)$  is likely to be a mess, since there's no simple rule for  $\ln(a + b)$ . *Your mileage will vary.*

## 8 Conclusions and a few final tidbits

With these tricks under your belt, you'll know just about as much about taking derivatives as I do. The only differences are that I also know a few multi-dimensional tricks that come up once in a blue moon, and that I've had a ton of practice. So get out there and practice!

Finally, look out for situations where you can simplify things. If you are taking the derivative of  $\sin(x)/x^2$ , you don't have to use the quotient rule, which is fairly ugly. Rewrite it as  $x^{-2} \sin(x)$  and use the product rule instead! If you have a quotient and the denominator is a constant (e.g., you're taking the derivative of  $\log_a(x) = \ln(x)/\ln(a)$ ), you don't need the quotient rule! The derivative of  $f/c$  is  $f'/c$ . You *could* use the quotient rule to get  $(cf' - f'c)/c^2 = cf'/c^2 = f'/c$  (since  $c' = 0$ ), but that's needlessly complicated. Constants just come along for the ride.

Good luck!

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